ABSTRACT FORWARD PROPAGATE

KILDALL'S ALGORITHM

AbstractForwardPropagate(Γ_c,P)
  S := $\{l_0\}$;
  $\hat{\mu}(l_0)$:= α(P);
  $\hat{\mu}(l)$:= $\perp$, for $l \in L \backslash \{l_0\}$;
  while S $\neq \varnothing$ do{
    l := Choose S;
    S := S \ $\{l\}$;
    foreach $(l, c, l') \in T$ do{
      F := $\hat{sp}(\hat{\mu}(l), c)$;
      if $\neg(\mathsf{F} \leq \hat{\mu}(l'))$ then{
        $\hat{\mu}(l') := \hat{\mu}(l') \sqcup F$;
        S := S $\cup \{l'\}$;
      }
    }
  }

- Does this algorithm actually calculate the abstract JOP?
- What are the conditions under which it is guaranteed to terminate?

A Unified Approach to Global Program Optimization. Gary Kildall. POPL 1973.

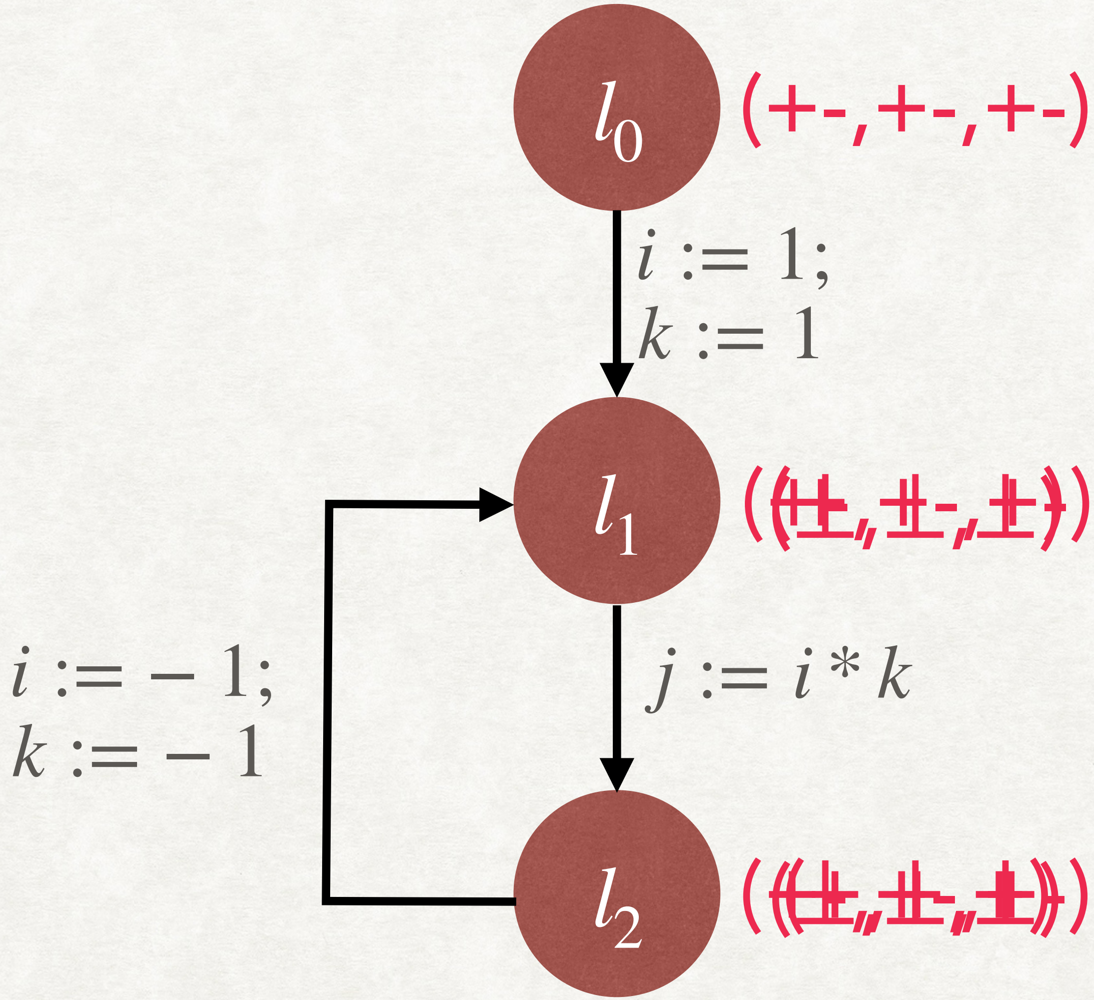
ABSTRACT FORWARD PROPAGATE

KILDALL'S ALGORITHM

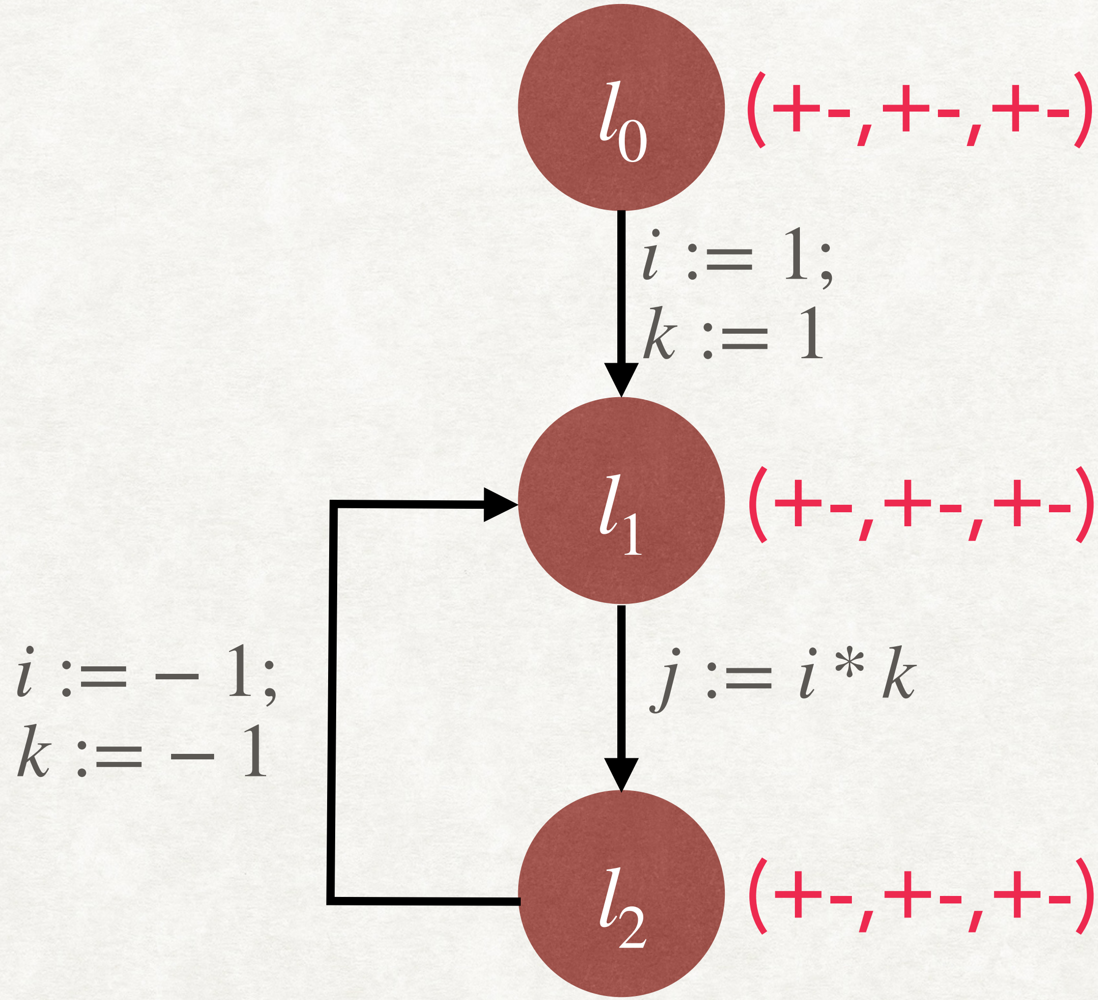
```
AbstractForwardPropagate(Γ_c,P)
  S := {l_0};
  μ̂_K(l_0) := α(P);
  μ̂_K(l) := ⊥, for l ∈ L\{l_0};
  while S ≠ ∅ do{
      l := Choose S;
      S := S \ {l};
      foreach (l,c,l') ∈ T do{
          F := f̂_c(μ̂_K(l));
          if ¬(F ≤ μ̂_K(l')) then{
              μ̂_K(l') := μ̂_K(l') ⊔ F;
              S := S ∪ {l'};
          }
      }
  }
```

A Unified Approach to Global Program Optimization. Gary Kildall. POPL 1973.

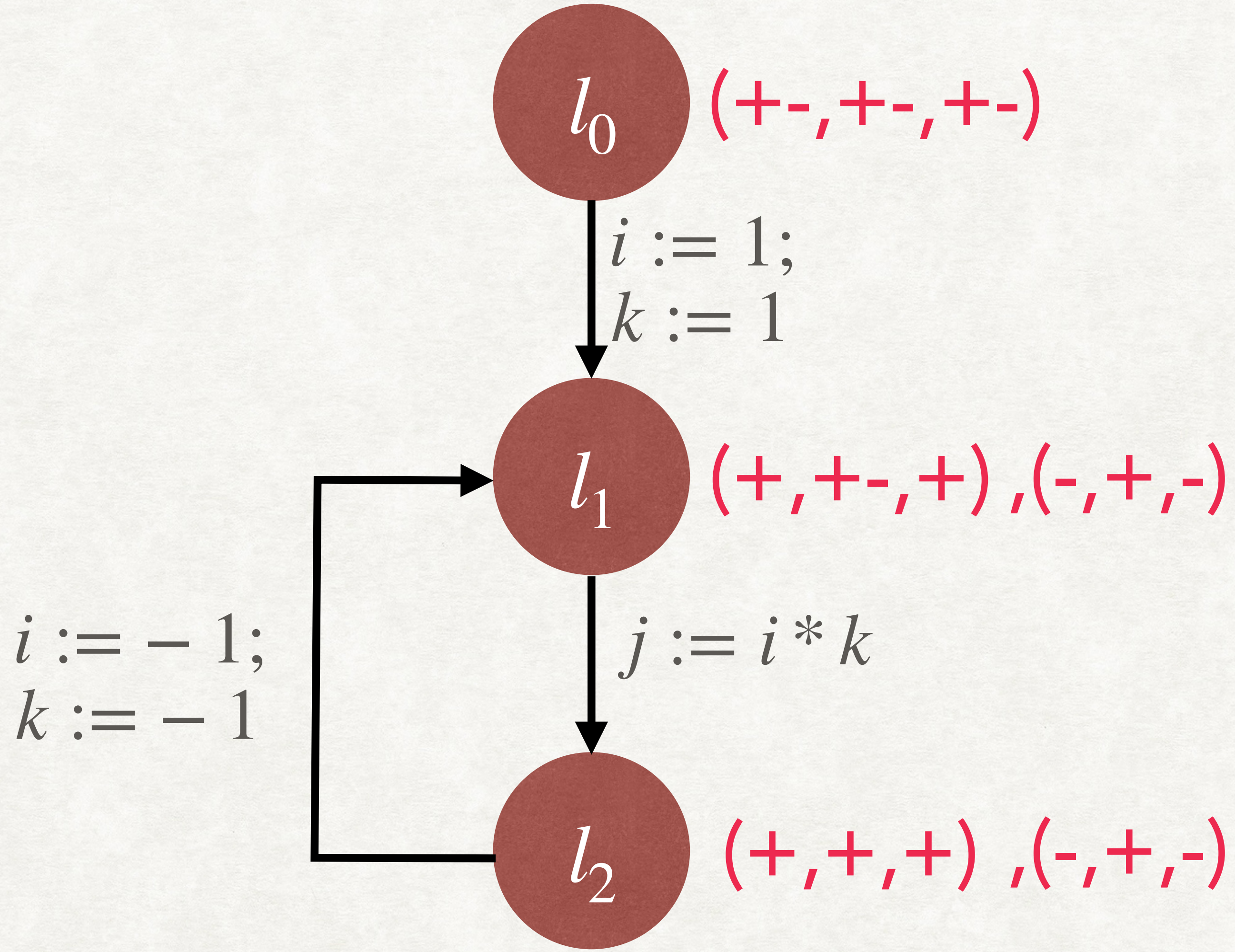
EXAMPLE - KILDALL'S ALGORITHM



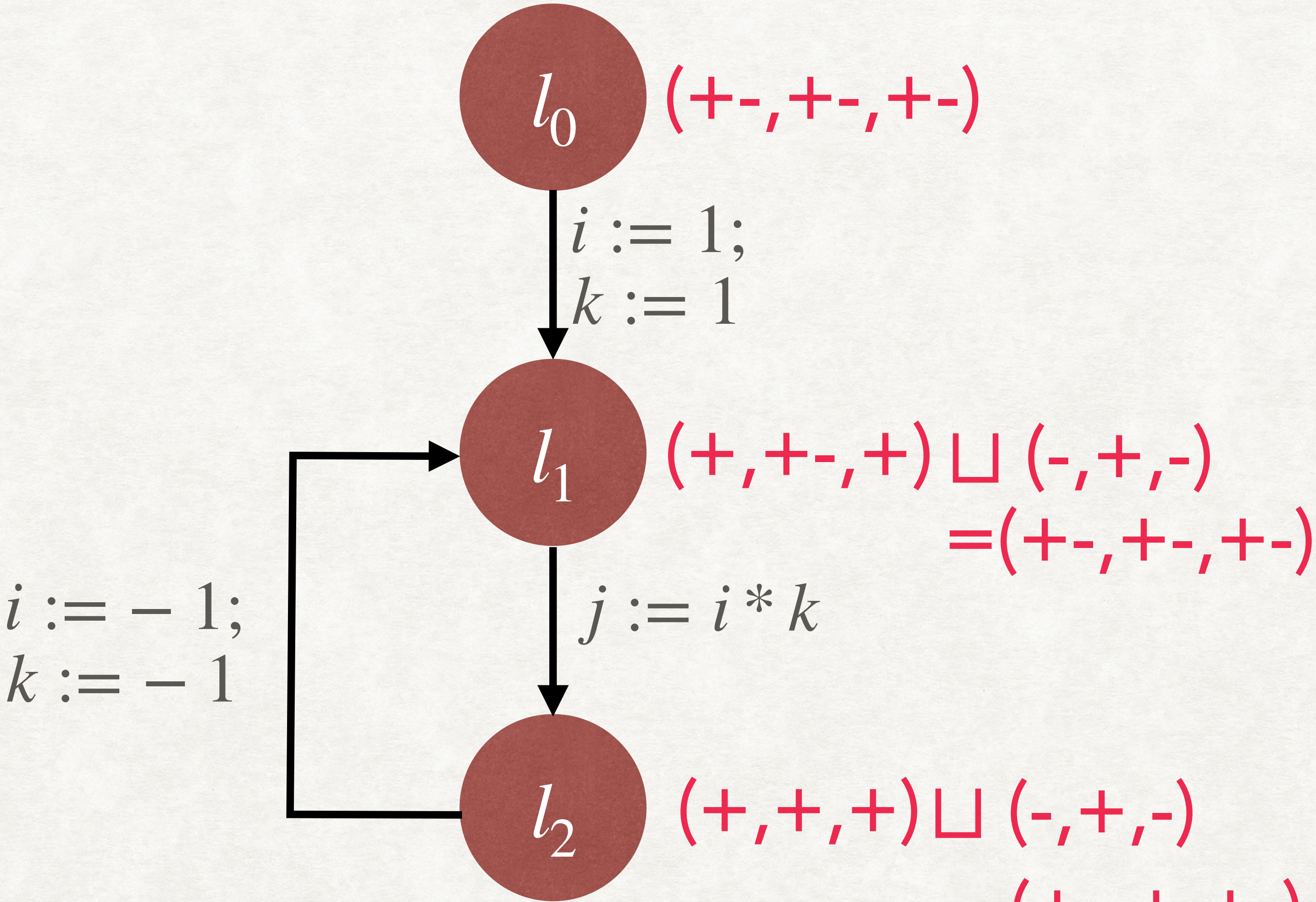
EXAMPLE - KILDALL'S ALGORITHM



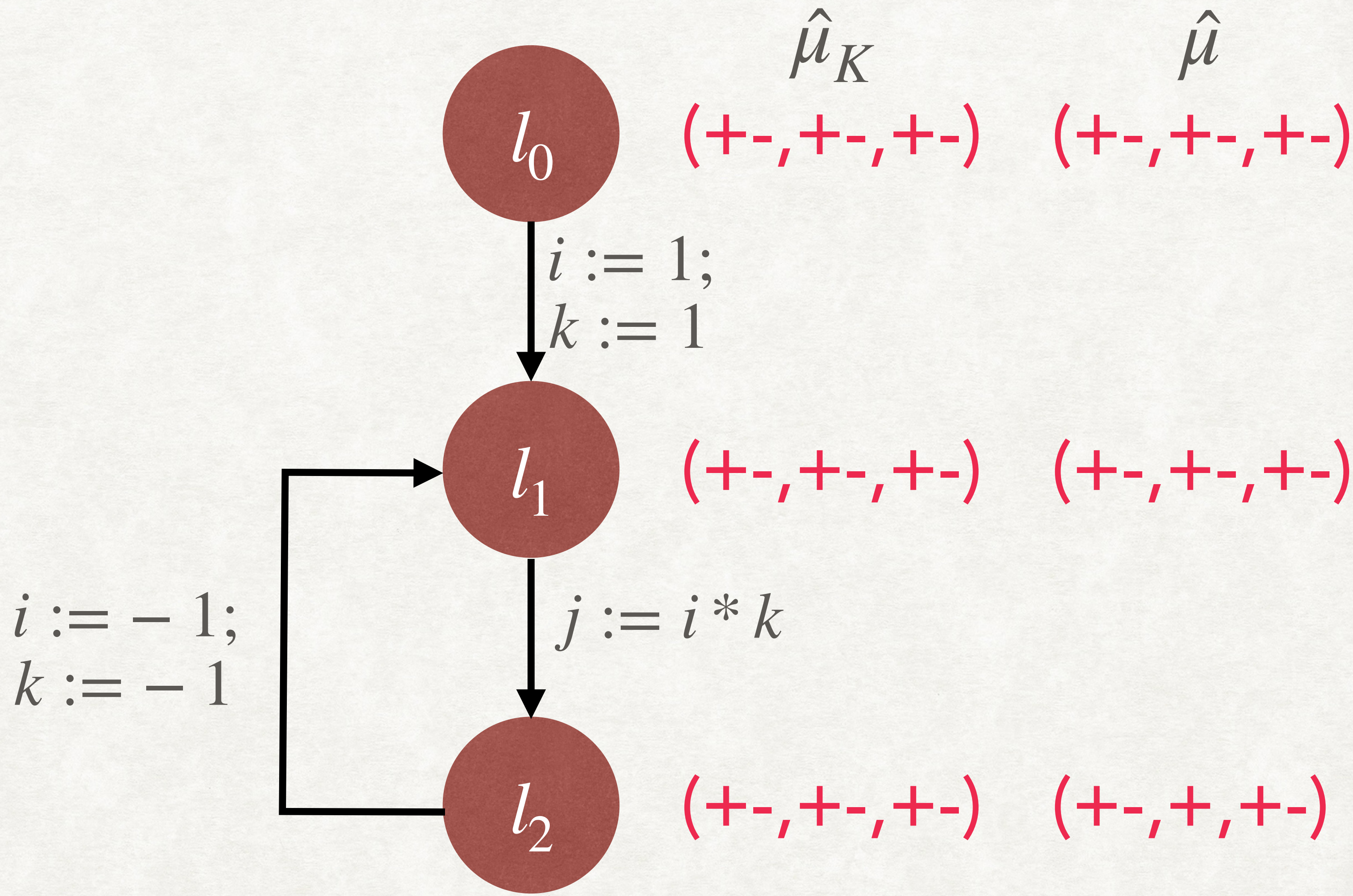
EXAMPLE - ABSTRACT JOP



EXAMPLE - ABSTRACT JOP



EXAMPLE - KILDALL VS ABSTRACT JOP



$\hat{\mu}_K \neq \hat{\mu}$: This is because Kildall's Algorithm applies join eagerly

We will prove that $\hat{\mu}_K \geq \hat{\mu}$

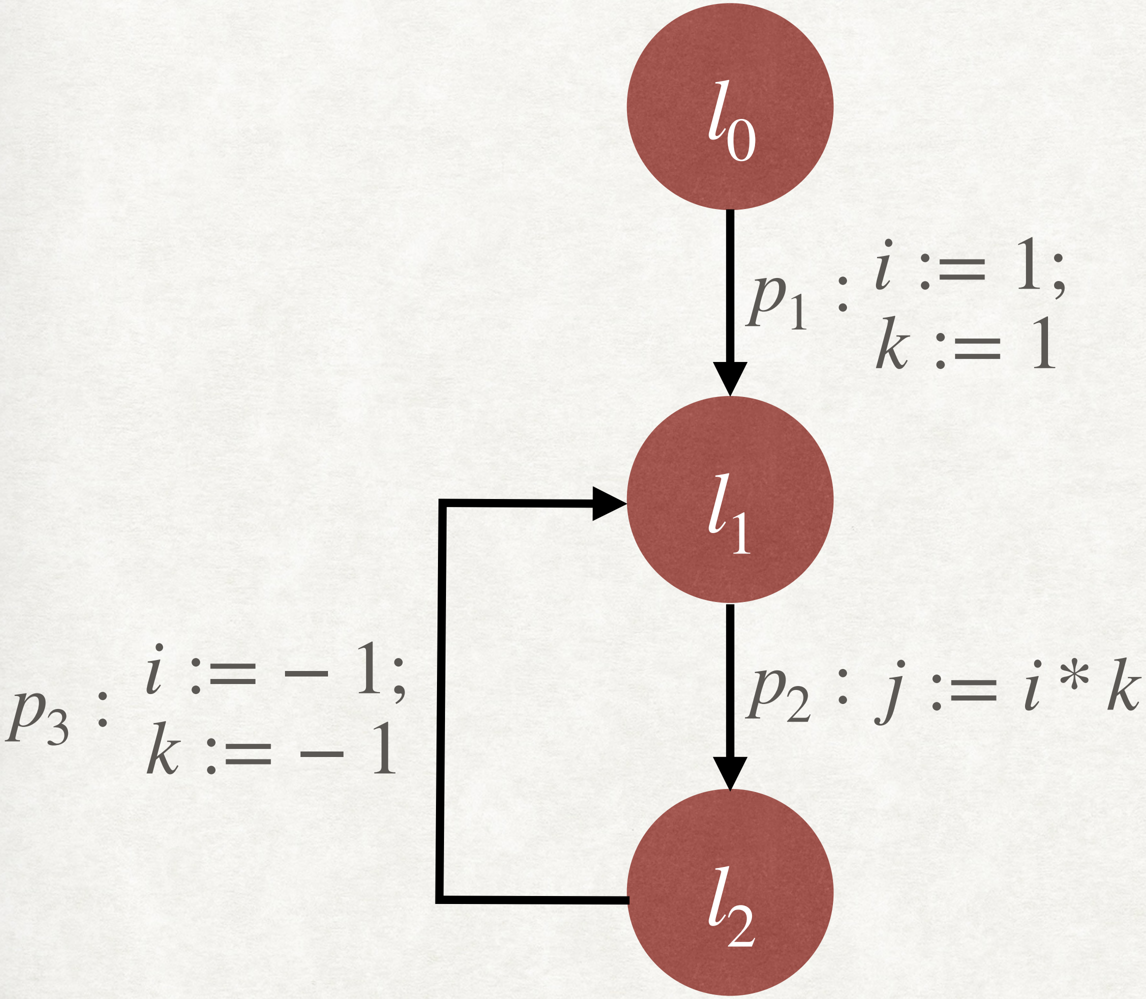
PROPERTIES OF KILDALL'S ALGORITHM

1. The values computed using Kildall's algorithm are an over-approximation of the abstract JOP, if the underlying AI framework is monotonic.
2. In general, Kildall's algorithm computes the least solution to a system of equations.
3. If the abstract domain satisfies the ascending chain condition, then Kildall's algorithm is guaranteed to terminate.

DATAFLOW EQUATIONS

- Program $\Gamma_c = (V, L, l_0, l_e, T)$ induces a system of data flow equations:
  - $X_{l_0} = d_0$
  - For all other locations $l \in L \backslash \{l_0\},\ X_l = \bigsqcup_{(l',c,l) \in T} \hat{f}_c(X_{l'})$
- For collecting semantics, replace d_0 with c_0, $\sqcup$ with $\cup$ and $\hat{f}_c$ with f_c.

EXAMPLE - DATAFLOW EQUATIONS



$$X_{l_0} = d_0$$

$$X_{l_1} = \hat{f}_{p_1}(X_{l_0}) \sqcup \hat{f}_{p_3}(X_{l_2})$$

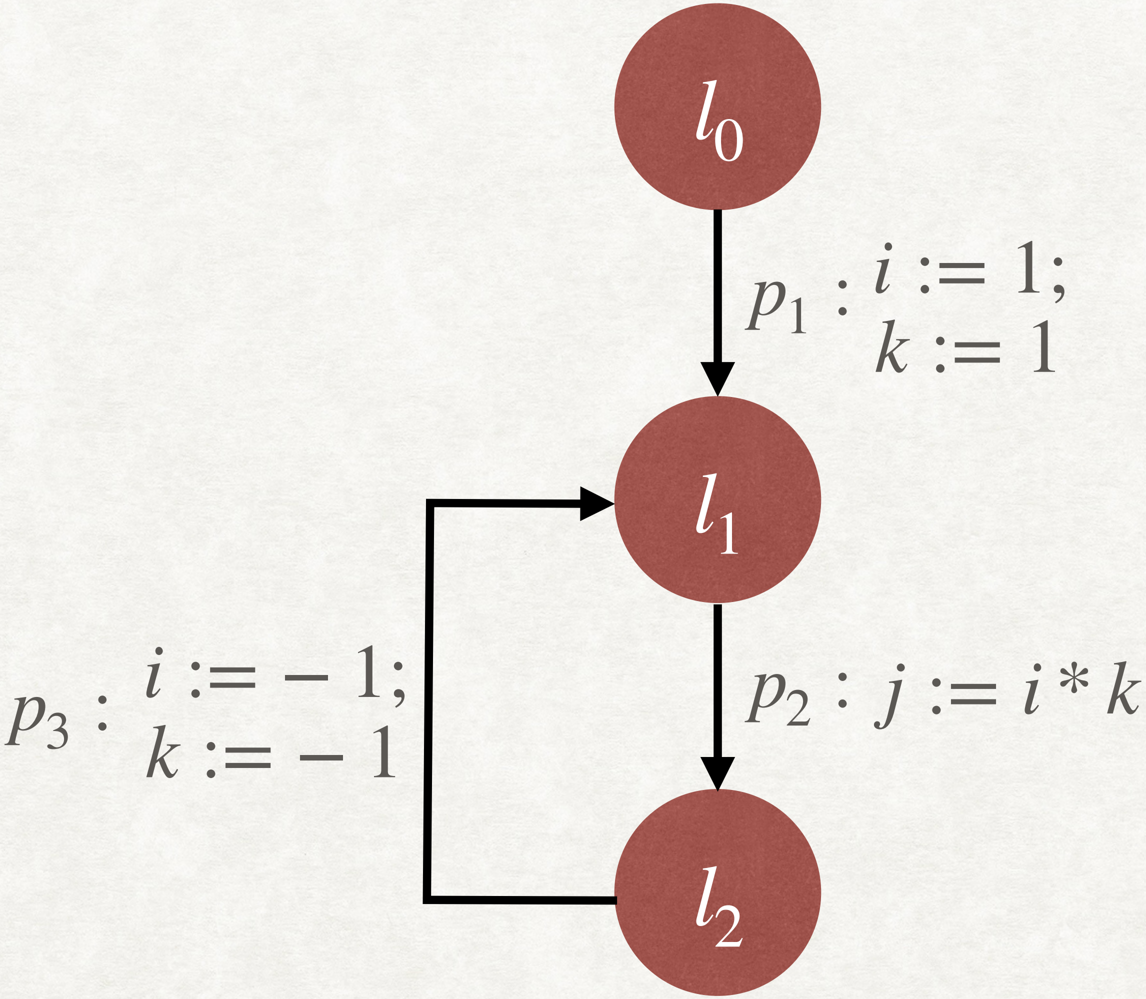
$$X_{l_2} = \hat{f}_{p_2}(X_{l_1})$$

DATAFLOW EQUATIONS AS FUNCTION

- Consider the 'vectorised' lattice $(\bar{D}, \bar{\leq})$, where $\bar{D} = D^{|L|}$.
  - $\bar{d} \,\bar{\leq}\, \bar{d}' \Leftrightarrow \forall l \in L \,.\, \bar{d}(l) \leq \bar{d}'(l)$
  - Homework: Prove that if $(D, \leq)$ is a complete lattice, then $(\bar{D}, \bar{\leq})$ is also a complete lattice.
- We can view the data flow equations as a function $\bar{f} : \bar{D} \rightarrow \bar{D}$:
  - $(\bar{f}(\bar{d}))(l_0) = d_0$
  - $(\bar{f}(\bar{d}))(l) = \bigsqcup_{(l',c,l) \in T} \hat{f}_c(\bar{d}(l'))$

DATAFLOW EQUATIONS AS FUNCTION

EXAMPLE



Notice that a fixpoint of $\bar{f}$ is a solution to the dataflow equations

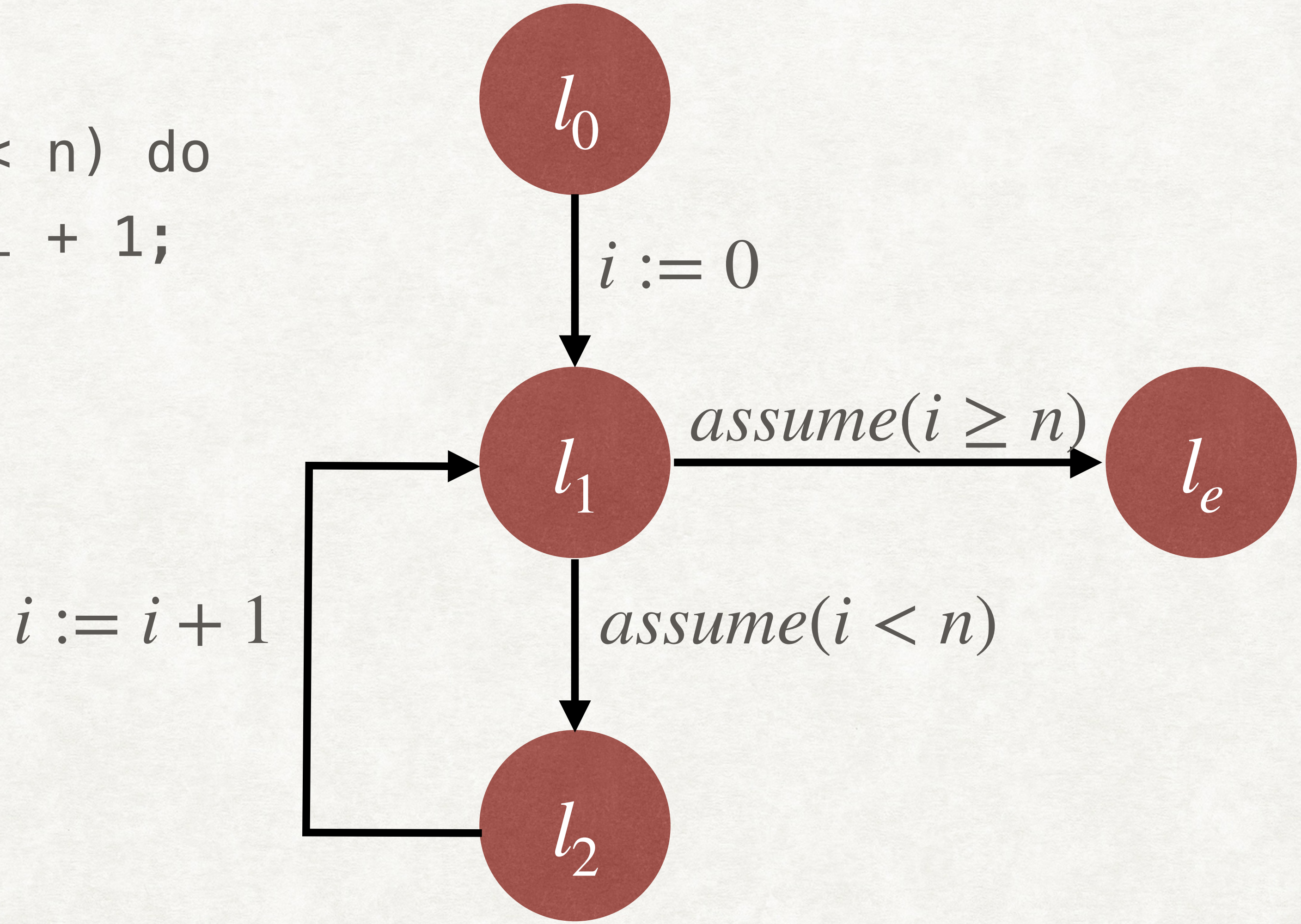
$$\bar{f}(d_{l_0}, d_{l_1}, d_{l_2}) = (d_0, \hat{f}_{p_1}(d_{l_0}) \sqcup \hat{f}_{p_3}(d_{l_2}), \hat{f}_{p_2}(d_{l_1}))$$

DATAFLOW EQUATIONS AS FUNCTION

- If every abstract transfer function $\hat{f} : D \to D$ is monotonic, then the function $\bar{f} : \bar{D} \to \bar{D}$ is also monotonic.
  - Homework: Prove this.
- We have a monotonic function $\bar{f}$ on a complete lattice $\bar{D}$. Hence, we can apply Knaster-Tarski fixpoint theorem.
- The least fixpoint $lfp(\bar{f})$ exists, and is in fact the least solution to the dataflow equations.
- We will show that Kildall's algorithm actually computes $lfp(\bar{f})$.
- Note that we can also use the sequence $\perp, \bar{f}(\perp), \bar{f}^2(\perp), \ldots$ to compute $lfp(\bar{f})$.
  - This method is also called Kleene Iteration.

LFP INTRODUCES THE LEAST OVER APPROXIMATION: EXAMPLE

```
i := 0;
while(i < n) do
    i := i + 1;
```



$(+-, +, +, +)$ is a solution to the data flow equations,

And $(+-, +-, +-, +-)$ is also another solution

ABSTRACT JOP $\leq lfp(\bar{f})$ FOR MONOTONIC AI FRAMEWORK

PROOF

- Given AI $(D, \leq, \alpha, \gamma, \hat{F}_D)$, if all functions in $\hat{F}_D$ are monotonic, then Abstract JOP $\leq lfp(\bar{f})$.

Proof: Abstract JOP $\hat{\mu}(l) = \bigsqcup_{\pi \in \Pi_l} \hat{f}_\pi(d_0)$

Let $lfp(\bar{f}) = \bar{d}$. We have to show that $\forall l \in L \,.\, \hat{\mu}(l) \leq \bar{d}(l)$.

We will show that for all locations l, all paths $\pi \in \Pi_l$, $\hat{f}_\pi(d_0) \leq \bar{d}(l)$.

This proves the required result. Why?

We will use induction on length of the paths.

Base Case: Paths π of length 0 are empty and end at l_0. Hence, $\hat{f}_\pi(d_0) = d_0$.

Since $\bar{f}(\bar{d}) = \bar{d}$ and $(\bar{f}(\bar{d}))(l_0) = d_0$, we have $\bar{d}(l_0) = d_0$.

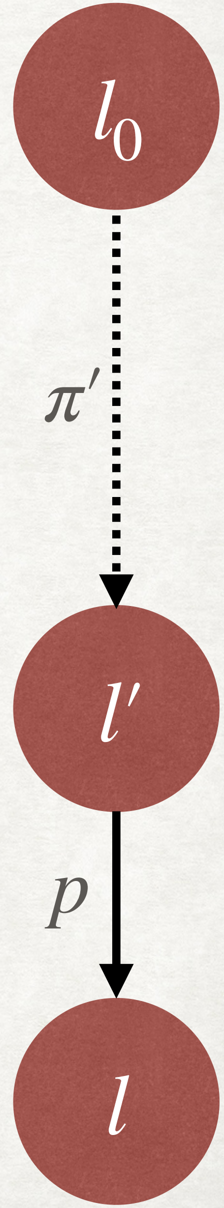
Thus, $\hat{f}_\pi(d_0) \leq \bar{d}(l_0)$

ABSTRACT JOP $\leq lfp(\bar{f})$ FOR MONOTONIC AI FRAMEWORK

PROOF

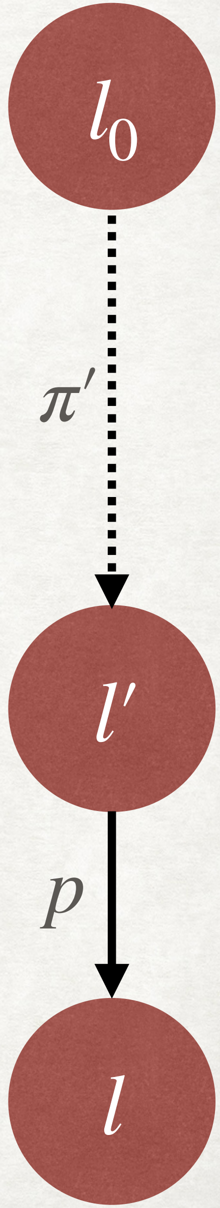
Inductive Case: Assume that the claim holds for all paths of length n.

Consider a path π of length $n+1$ ending at location l.



ABSTRACT JOP $\leq lfp(\bar{f})$ FOR MONOTONIC AI FRAMEWORK

PROOF



Inductive Case: Assume that the claim holds for all paths of length n.

Consider a path π of length $n+1$ ending at location l.

Let π' be the prefix of the path of length n, ending at location l'.

By Inductive Hypothesis, $\hat{f}_{\pi'}(d_0) \leq \bar{d}(l')$.

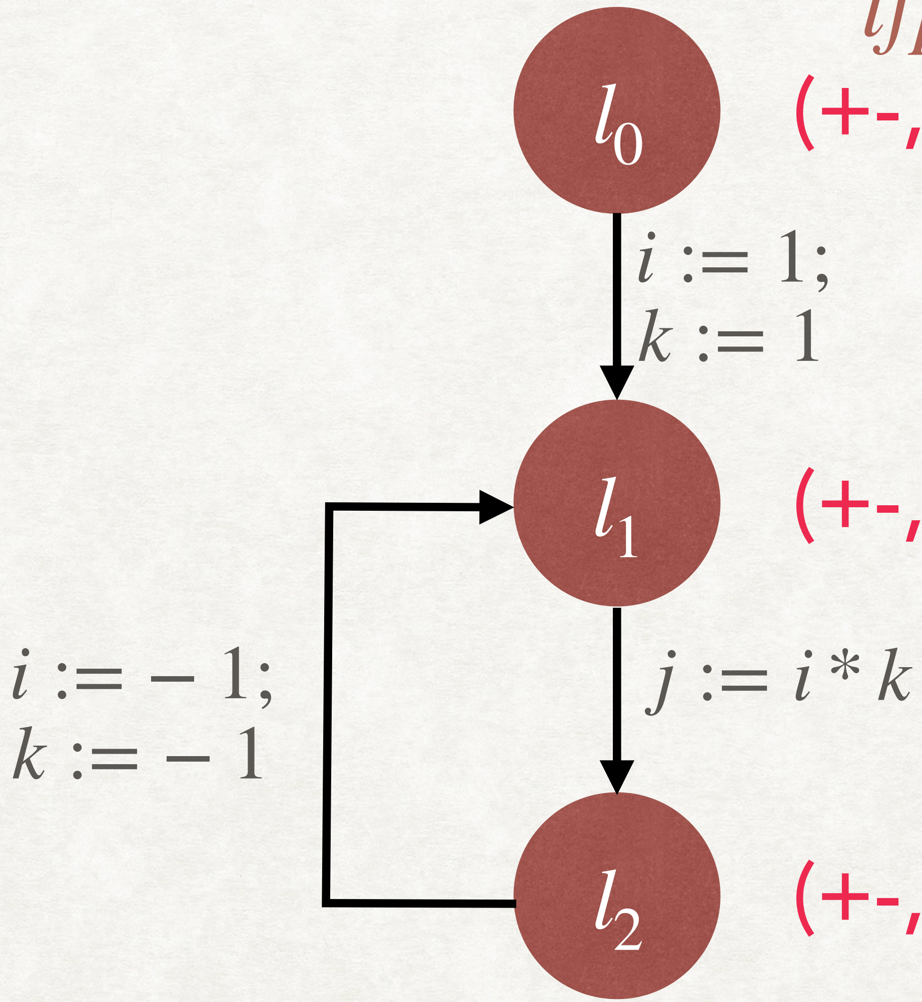
Since $\hat{f}_p$ is monotonic, $\hat{f}_p(\hat{f}_{\pi'}(d_0)) \leq \hat{f}_p(\bar{d}(l'))$.

Hence, $\hat{f}_\pi(d_0) \leq \hat{f}_p(\bar{d}(l'))$.

Now $\bar{f}(\bar{d}) = \bar{d}$. Hence, $\bar{d}(l) = \bigsqcup_{(l',p,l)\in T} \hat{f}_p(\bar{d}(l'))$.

Hence, $\hat{f}_p(\bar{d}(l')) \leq \bar{d}(l)$. Thus, $\hat{f}_\pi(d_0) \leq \bar{d}(l)$.

EXAMPLE - LFP VS ABSTRACT JOP



	$lfp(\bar{f})$	Abstract JOP
l_0	(+-,+-,+-)	(+-,+-,+-)
l_1	(+-,+-,+-)	(+-,+-,+-)
l_2	(+-,+-,+-)	(+-,+,+-)

$$\bar{f}(d_{l_0}, d_{l_1}, d_{l_2}) = (d_0, \hat{f}_{p_1}(d_{l_0}) \sqcup \hat{f}_{p_3}(d_{l_2}), \hat{f}_{p_2}(d_{l_1}))$$

DISTRIBUTIVE AND INFINITELY DISTRIBUTIVE FUNCTIONS

- Given two posets $(D_1, \leq_1)$ and $(D_2, \leq_2)$, function $f : D_1 \to D_2$ is called distributive if for $x, y \in D_1$ such that $x \sqcup_1 y$ exists, then $f(x) \sqcup_2 f(y)$ also exists, and $f(x \sqcup_1 y) = f(x) \sqcup_2 f(y)$.
- Given two posets $(D_1, \leq_1)$ and $(D_2, \leq_2)$, function $f : D_1 \to D_2$ is called infinitely distributive if for all $X \subseteq D_1$ such that $\sqcup_1 X$ exists, then $\sqcup_2 f(X)$ also exists, and $\sqcup_2 f(X) = f(\sqcup_1 X)$.
- Exercise: If f is distributive, then f is also monotonic.

ABSTRACT JOP = $lfp(\bar{f})$ FOR INFINITELY DISTRIBUTIVE AI FRAMEWORK

PROOF

- Given AI $(D, \leq, \alpha, \gamma, \hat{F}_D)$, if all functions in $\hat{F}_D$ are infinitely distributive, then Abstract JOP = $lfp(\bar{f})$.

Proof: We will show that Abstract JOP ($\hat{\mu}$) is a fixpoint of $\bar{f}$. This is sufficient to prove the result. Why?

ABSTRACT JOP = $lfp(\bar{f})$ FOR INFINITELY DISTRIBUTIVE AI FRAMEWORK

PROOF

- Given AI $(D, \leq, \alpha, \gamma, \hat{F}_D)$, if all functions in $\hat{F}_D$ are infinitely distributive, then Abstract JOP = $lfp(\bar{f})$.

Proof: We will show that Abstract JOP ($\hat{\mu}$) is a fixpoint of $\bar{f}$. This is sufficient to prove the result. Why?

$$
\begin{aligned}
(\bar{f}(\hat{\mu}))(l) &= \bigsqcup_{(l',p,l)\in T} \hat{f}_p(\hat{\mu}(l')) \\
&= \bigsqcup_{(l',p,l)\in T} \hat{f}_p\Big(\bigsqcup_{\pi\in\Pi_{l'}} \hat{f}_\pi(d_0)\Big) \\
&= \bigsqcup_{(l',p,l)\in T} \bigsqcup_{\pi\in\Pi_{l'}} \hat{f}_p \circ \hat{f}_\pi(d_0)
\end{aligned}
$$

ABSTRACT JOP = $lfp(\bar{f})$ FOR INFINITELY DISTRIBUTIVE AI FRAMEWORK

PROOF

$$(\bar{f}(\hat{\mu}))(l) = \bigsqcup_{(l',p,l)\in T} \bigsqcup_{\pi\in\Pi_{l'}} \hat{f}_p \circ \hat{f}_\pi(d_0)$$

And we know that $\hat{\mu}(l) = \bigsqcup_{\pi'\in\Pi_l} \hat{f}_{\pi'}(d_0)$.

Then, due to associativity of $\sqcup$, $(\bar{f}(\hat{\mu}))(l) = \hat{\mu}(l)$.

Thus, $\hat{\mu}$ is a fixpoint of $\bar{f}$. We know from previous result that $\hat{\mu} \leq lfp(\bar{f})$. Thus, $\hat{\mu} = lfp(\bar{f})$.

RECALL: SIGN ABSTRACT DOMAIN

- The abstract transfer functions in sign abstract domain are monotonic, but not infinitely distributive.

Consider p : j := i*k and $d_1 = (+, +-, +)$, $d_2 = (-, +-, -)$.

Then, $\hat{f}_p(d_1 \sqcup d_2) = ???$

RECALL: SIGN ABSTRACT DOMAIN

- The abstract transfer functions in sign abstract domain are monotonic, but not infinitely distributive.

Consider p : j := i*k and $d_1 = (\,+\,,\,+\,-\,,\,+\,)$, $d_2 = (\,-\,,\,+\,-\,,\,-\,)$.

Then, $\hat{f}_p(d_1 \sqcup d_2) = (\,+\,-\,,\,+\,-\,,\,+\,-\,)$.

RECALL: SIGN ABSTRACT DOMAIN

- The abstract transfer functions in sign abstract domain are monotonic, but not infinitely distributive.

Consider p : j := i*k and $d_1 = (+ , + - , +)$, $d_2 = (- , + - , -)$.

Then, $\hat{f}_p(d_1 \sqcup d_2) = (+ - , + - , + -)$.

$\hat{f}_p(d_1) \sqcup \hat{f}_p(d_2) = ???$

RECALL: SIGN ABSTRACT DOMAIN

- The abstract transfer functions in sign abstract domain are monotonic, but not infinitely distributive.

Consider p : j := i*k and $d_1 = (+, +-, +)$, $d_2 = (-, +-, -)$.

Then, $\hat{f}_p(d_1 \sqcup d_2) = (+-, +-, +-)$.

$\hat{f}_p(d_1) \sqcup \hat{f}_p(d_2) = (+, +, +) \sqcup (-, +, -) = (+-, +, +-)$

RECALL: SIGN ABSTRACT DOMAIN

- The abstract transfer functions in sign abstract domain are monotonic, but not infinitely distributive.

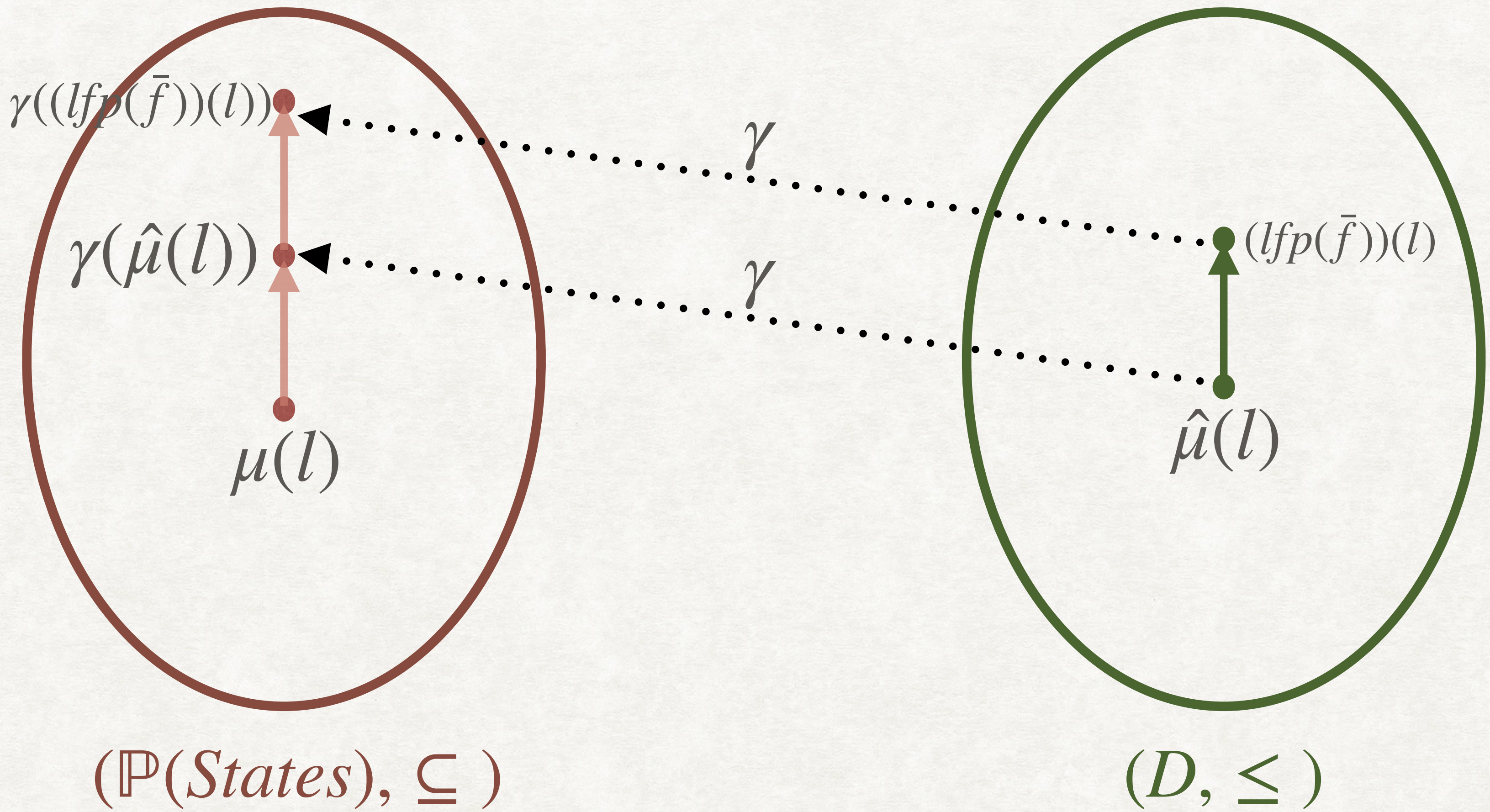
Consider p : j := i*k and $d_1 = (+\,, +-\,, +)$, $d_2 = (-\,, +-\,, -)$.

Then, $\hat{f}_p(d_1 \sqcup d_2) = (+-\,, +-\,, +-)$.

$\hat{f}_p(d_1) \sqcup \hat{f}_p(d_2) = (+\,, +\,, +) \sqcup (-\,, +\,, -) = (+-\,, +\,, +-)$

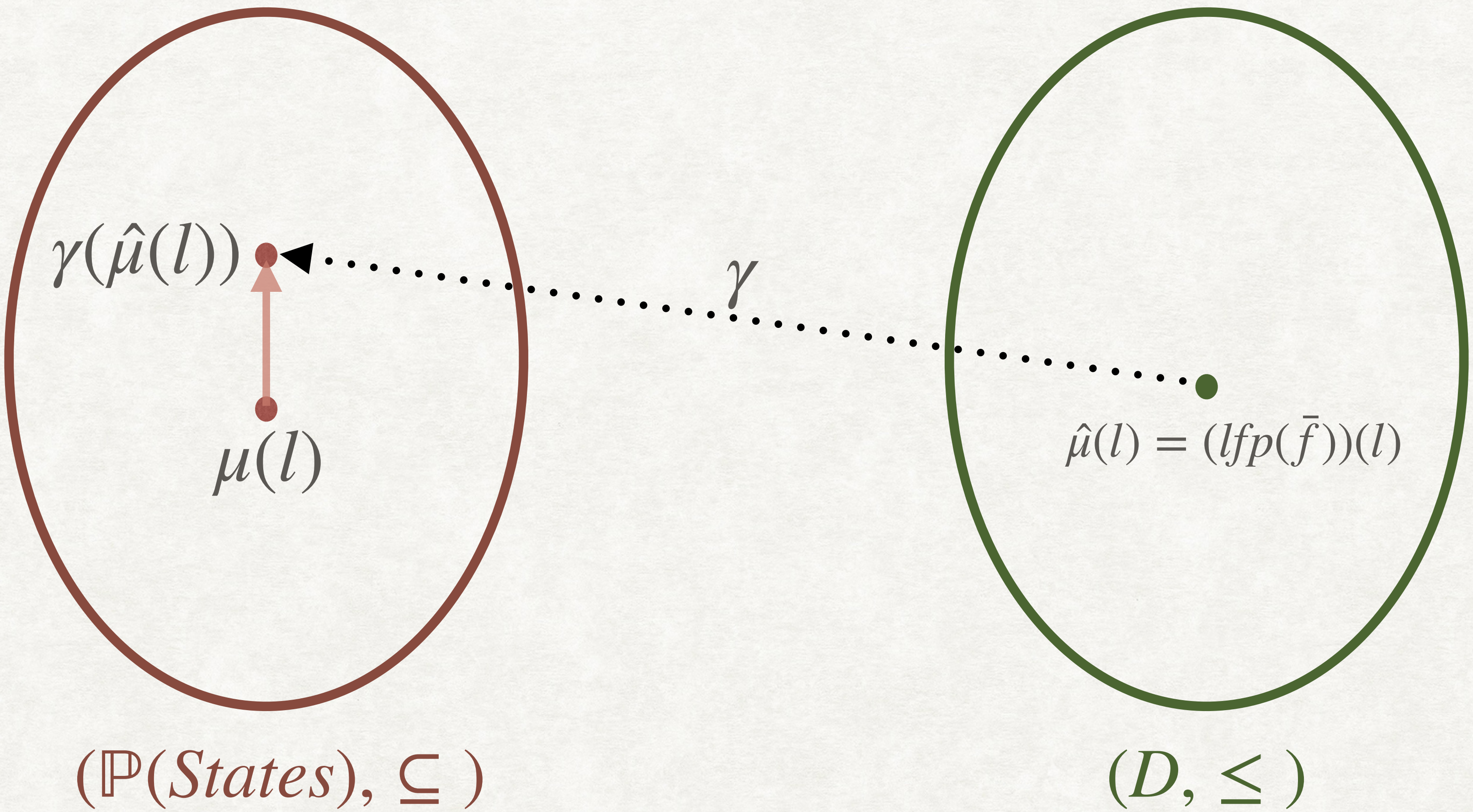
- The concrete transfer functions are always infinitely distributive. Hence, the concrete JOP is the least solution of the concrete data-flow equations.

BIG PICTURE



For Monotonic AI Framework

BIG PICTURE



For Infinitely Distributive AI Framework

KILDALL'S ALGORITHM COMPUTES $lfp(\bar{f})$

PROOF

- First, we will show that $\hat{\mu}_K \leq lfp(\bar{f})$

We will show that $\hat{\mu}_K \leq lfp(\bar{f})$ is a loop invariant of the outer while loop.

At the beginning, $\hat{\mu}_K(l_0) = \alpha(P) \leq d_0$.

Hence, $\forall l \,.\, \hat{\mu}_K(l) \leq (lfp(\bar{f}))(l)$.

Assuming that the claim holds at the beginning of some iteration, let $\hat{\mu}_K = \bar{d}$, $lfp(\bar{f}) = \bar{g}$. We have $\bar{d} \leq \bar{g}$.

```
AbstractForwardPropagate(Γ_c,P)
  S := {l_0};
  μ̂_K(l_0) := α(P);
  μ̂_K(l) := ⊥, for l ∈ L\{l_0};
  while S ≠ ∅ do{
     l := Choose S;
     S := S \ {l};
     foreach (l,c,l') ∈ T do{
        F := f̂_c(μ̂_K(l));
        if ¬(F ≤ μ̂_K(l')) then{
           μ̂_K(l') := μ̂_K(l') ⊔ F;
           S := S ∪ {l'};
        }
     }
  }
```

KILDALL'S ALGORITHM COMPUTES $lfp(\bar{f})$

PROOF

```
AbstractForwardPropagate(Γ_c,P)
  S := {l_0};
  μ̂_K(l_0) := α(P);
  μ̂_K(l) := ⊥, for l ∈ L\{l_0};
  while S ≠ ∅ do{
    l := Choose S;
    S := S \ {l};
    foreach (l,c,l') ∈ T do{
      F := f̂_c(μ̂_K(l));
      if ¬(F ≤ μ̂_K(l')) then{
        μ̂_K(l') := μ̂_K(l') ⊔ F;
        S := S ∪ {l'};
      }
    }
  }
```

KILDALL'S ALGORITHM COMPUTES $lfp(\bar{f})$

PROOF

For some successor l' of l,
$\hat{\mu}_K(l') = d(l') \sqcup \hat{f}_c(d(l))$.

Now, $\bar{d}(l) \leq \bar{g}(l) \Rightarrow \hat{f}_c(\bar{d}(l)) \leq \hat{f}_c(\bar{g}(l))$.

Further, $\bar{g}(l') = \displaystyle\bigsqcup_{(l,c,l')\in T} \hat{f}_c(\bar{g}(l))$

Hence, $\bar{g}(l') \geq \hat{f}_c(\bar{g}(l)) \geq \hat{f}_c(\bar{d}(l))$

We also know that $\bar{g}(l') \geq \bar{d}(l')$.

Thus, $\bar{g}(l') \geq \bar{d}(l') \sqcup \hat{f}_c(\bar{d}(l))$.

Hence, $\bar{g}(l') \geq \hat{\mu}_K(l')$.

```
AbstractForwardPropagate(Γ_c,P)
  S := {l_0};
  μ̂_K(l_0) := α(P);
  μ̂_K(l) := ⊥, for l ∈ L\{l_0};
  while S ≠ ∅ do{
    l := Choose S;
    S := S \ {l};
    foreach (l,c,l') ∈ T do{
      F := f̂_c(μ̂_K(l));
      if ¬(F ≤ μ̂_K(l')) then{
        μ̂_K(l') := μ̂_K(l') ⊔ F;
        S := S ∪ {l'};
      }
    }
  }
```

KILDALL'S ALGORITHM COMPUTES $lfp(\bar{f})$

PROOF

Next, we will show that $\hat{\mu}_K \geq lfp(\bar{f})$.

To prove this, we will show that when the algorithm terminates, the final $\hat{\mu}_K$ is a post-fixpoint of $\bar{f}$, i.e. $\bar{f}(\hat{\mu}_K) \leq \hat{\mu}_K$.

Then, by Knaster-Tarski theorem, $lfp(\bar{f})$ is the glb of all post-fixpoints, and hence the claim follows.

We will prove that following is a loop invariant of the outer while-loop:

$$\forall l \in L \backslash S \,.\, \forall l' \in L \,.\, (l, c, l') \in T \Rightarrow \hat{\mu}_K(l') \geq \hat{f}_c(\hat{\mu}_K(l))$$

```
AbstractForwardPropagate(Γ_c,P)
  S := {l_0};
  μ̂_K(l_0) := α(P);
  μ̂_K(l) := ⊥, for l ∈ L\{l_0};
  while S ≠ ∅ do{
     l := Choose S;
     S := S \ {l};
     foreach (l,c,l') ∈ T do{
        F := f̂_c(μ̂_K(l));
        if ¬(F ≤ μ̂_K(l')) then{
           μ̂_K(l') := μ̂_K(l') ⊔ F;
           S := S ∪ {l'};
        }
     }
  }
```

KILDALL'S ALGORITHM COMPUTES $lfp(\bar{f})$

PROOF

$$\forall l \in L \backslash S \,.\, \forall l' \in L \,.\, (l, c, l') \in T$$
$$\Rightarrow \hat{\mu}_K(l') \geq \hat{f}_c(\hat{\mu}_K(l))$$

```
AbstractForwardPropagate(Γ_c,P)
  S := {l_0};
  μ̂_K(l_0) := α(P);
  μ̂_K(l) := ⊥, for l ∈ L\{l_0};
  while S ≠ ∅ do{
      l := Choose S;
      S := S \ {l};
      foreach (l,c,l') ∈ T do{
          F := f̂_c(μ̂_K(l));
          if ¬(F ≤ μ̂_K(l')) then{
              μ̂_K(l') := μ̂_K(l') ⊔ F;
              S := S ∪ {l'};
          }
      }
  }
```

KILDALL'S ALGORITHM COMPUTES $lfp(\bar{f})$

PROOF

$$\forall l \in L \backslash S \,.\, \forall l' \in L \,.\, (l, c, l') \in T$$

$$\Rightarrow \hat{\mu}_K(l') \geq \hat{f}_c(\hat{\mu}_K(l))$$

On exiting the loop, we will have

$$\forall l, l' \in L \,.\, (l, c, l') \in T \Rightarrow \hat{\mu}_K(l') \geq \hat{f}_c(\hat{\mu}_K(l))$$

$$\Rightarrow \forall l' \in L \,.\, \hat{\mu}_K(l') \geq \bigsqcup_{(l,c,l') \in T} \hat{f}_c(\hat{\mu}_K(l))$$

$$\Rightarrow \forall l' \in L \,.\, \hat{\mu}_K(l') \geq (\bar{f}(\hat{\mu}_K))(l')$$

```
AbstractForwardPropagate(Γ_c,P)
  S := {l_0};
  μ̂_K(l_0) := α(P);
  μ̂_K(l) := ⊥, for l ∈ L\{l_0};
  while S ≠ ∅ do{
      l := Choose S;
      S := S \ {l};
      foreach (l,c,l') ∈ T do{
          F := f̂_c(μ̂_K(l));
          if ¬(F ≤ μ̂_K(l')) then{
              μ̂_K(l') := μ̂_K(l') ⊔ F;
              S := S ∪ {l'};
          }
      }
  }
```

KILDALL'S ALGORITHM COMPUTES $lfp(\bar{f})$

PROOF

$$\forall l \in L \backslash S \,.\, \forall l' \in L \,.\, (l, c, l') \in T \Rightarrow \hat{\mu}_K(l') \geq \hat{f}_c(\hat{\mu}_K(l))$$

At the beginning, the invariant holds, assuming that $\hat{f}_c(\perp) = \perp$.

Note that if $\hat{f}_c(\perp) \neq \perp$, we can initialise S with L.

```
AbstractForwardPropagate(Γ_c,P)
  S := {l_0};
  μ̂_K(l_0) := α(P);
  μ̂_K(l) := ⊥, for l ∈ L\{l_0};
  while S ≠ ∅ do{
     l := Choose S;
     S := S \ {l};
     foreach (l,c,l') ∈ T do{
        F := f̂_c(μ̂_K(l));
        if ¬(F ≤ μ̂_K(l')) then{
           μ̂_K(l') := μ̂_K(l') ⊔ F;
           S := S ∪ {l'};
        }
     }
  }
```

KILDALL'S ALGORITHM COMPUTES $lfp(\bar{f})$

PROOF

$$\forall l \in L \backslash S \,.\, \forall l' \in L \,.\, (l, c, l') \in T \Rightarrow \hat{\mu}_K(l') \geq \hat{f}_c(\hat{\mu}_K(l))$$

Assume that the claim holds at the beginning of some iteration.

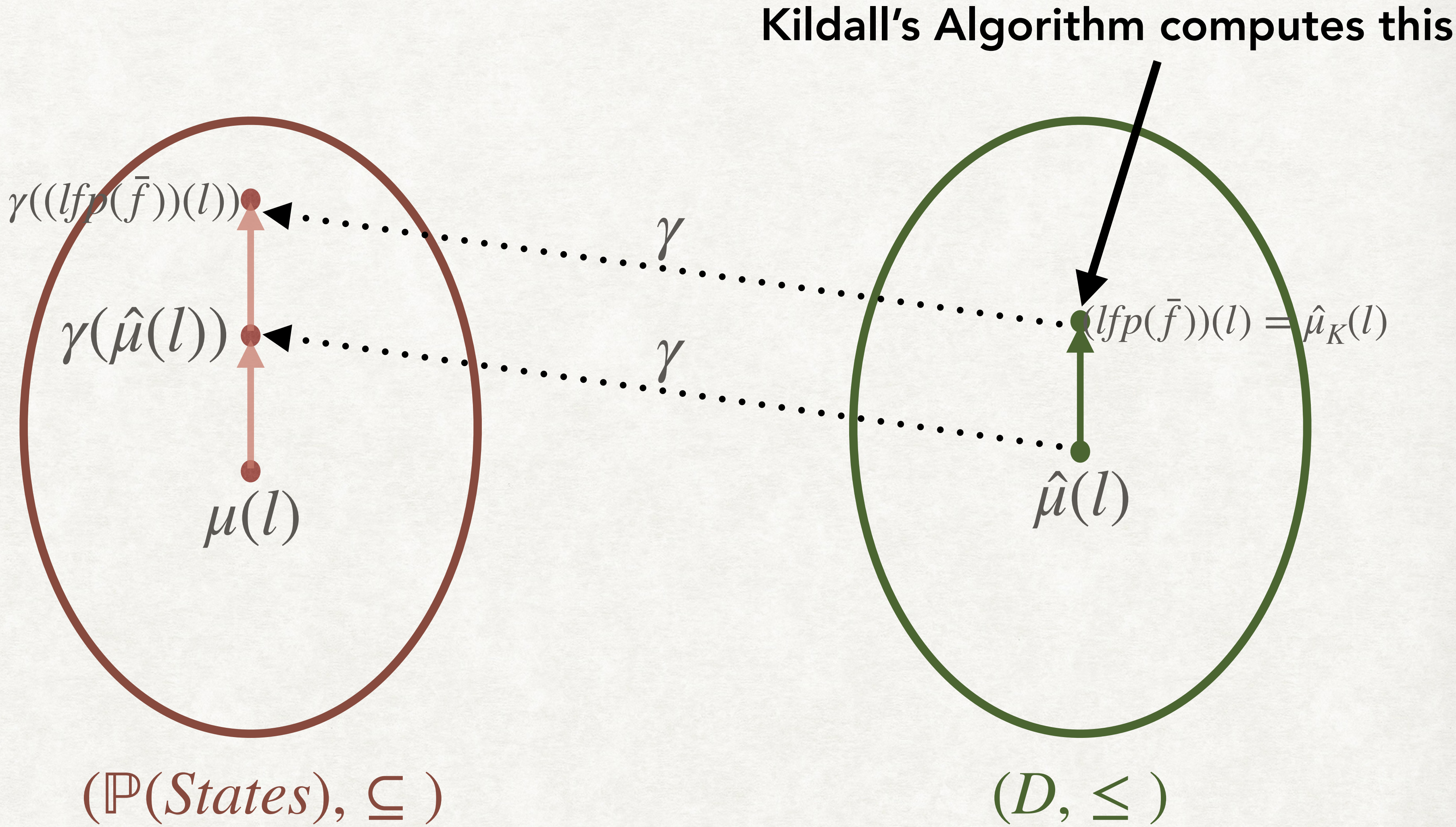
For each successor l' of l, either $\hat{\mu}_K(l') \geq \hat{f}_c(\hat{\mu}_K(l))$, or we enter the if-body and re-assign $\hat{\mu}_K(l')$ to ensure that $\hat{\mu}_K(l') \geq \hat{f}_c(\hat{\mu}_K(l))$.

Thus, the loop invariant continues to hold.

This concludes the proof that the final $\hat{\mu}_K = lfp(\bar{f})$.

```
AbstractForwardPropagate(Γc,P)
  S := {l0};
  μ̂K(l0) := α(P);
  μ̂K(l) := ⊥, for l ∈ L\{l0};
  while S ≠ ∅ do{
     l := Choose S;
     S := S \ {l};
     foreach (l,c,l') ∈ T do{
        F := f̂c(μ̂K(l));
        if ¬(F ≤ μ̂K(l')) then{
           μ̂K(l') := μ̂K(l') ⊔ F;
           S := S ∪ {l'};
        }
     }
  }
```

BIG PICTURE



For Monotonic AI Framework

KILDALL'S ALGORITHM: TERMINATION

- Consider the vector of values maintained by the algorithm across locations.
- After each iteration of the outer loop, either this vector increases or it stays the same and S decreases.
- If $(D, \leq)$ satisfies the ascending chain condition, then so does $(\bar{D}, \bar{\leq})$.
  - In this case, the loop is guaranteed to terminate.

```
AbstractForwardPropagate(Γ_c,P)
  S := {l_0};
  μ̂_K(l_0) := α(P);
  μ̂_K(l) := ⊥, for l ∈ L\{l_0};
  while S ≠ ∅ do{
      l := Choose S;
      S := S \ {l};
      foreach (l,c,l') ∈ T do{
          F := f̂_c(μ̂_K(l));
          if ¬(F ≤ μ̂_K(l')) then{
              μ̂_K(l') := μ̂_K(l') ⊔ F;
              S := S ∪ {l'};
          }
      }
  }
```

KILDALL'S ALGORITHM

SUFFICIENT CONDITIONS

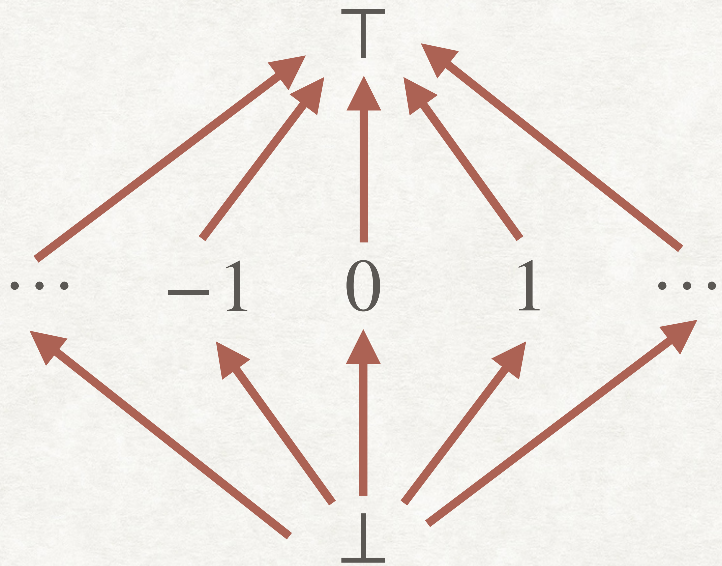
- Kildall's Algorithm can be used with an abstract domain $(D, \leq, \alpha, \gamma, \hat{F}_D)$ if:
  - $(D, \leq)$ is a complete lattice.
  - $(\mathbb{P}(State), \subseteq) \underset{\gamma}{\overset{\alpha}{\rightleftarrows}} (D, \leq)$
  - Every abstract transfer function in $\hat{F}_D$ is a consistent abstraction of the corresponding concrete transfer function.
  - Every abstract transfer function in $\hat{F}_D$ is monotonic.
  - $(D, \leq)$ satisfies the ascending chain condition.

APPLYING KILDALL'S ALGORITHM USING CONCRETE PROGRAM STATES

- Recall the concrete lattice of program states: $(\mathbb{P}(States), \subseteq)$ where $States = Var \rightarrow \mathbb{Z}$.
- Does this lattice satisfy ACC?
- Kildall's Algorithm using concrete lattice ≡ `ForwardPropagate` Algorithm .
  - Since the concrete lattice does not satisfy ACC, termination of Kildall's Algorithm is not guaranteed.
- Since the concrete transfer functions are infinitely distributive, LFP = JOP.

CONSTANT ABSTRACT DOMAIN

- $I = \mathbb{Z} \cup \{\top, \bot\}$
  - $\forall n \in \mathbb{Z} \,.\, \bot \leq n \leq \top$
  - Flat, but infinite lattice.
  - Satisfies ACC.
- $D = V \to I$



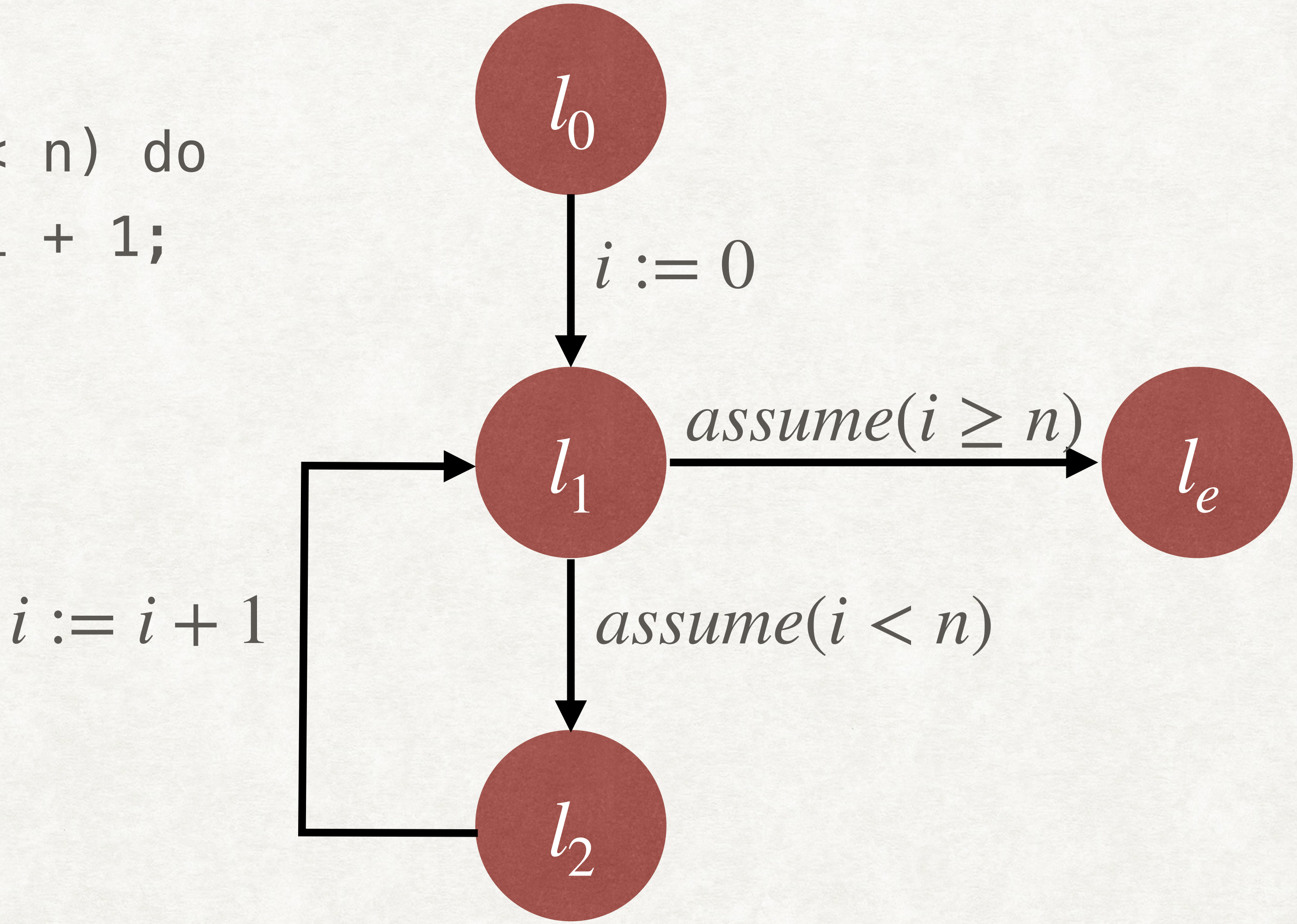
CONSTANT ABSTRACT DOMAIN

ABSTRACTION AND CONCRETIZATION FUNCTION

- $\alpha(c) = d$
  - If $c = \emptyset$, then $\forall v \,.\, d(v) = \bot$
  - Otherwise, $d(v) = \begin{cases} n & \text{if } \forall \alpha \in c \,.\, \sigma(v) = n \\ \top & \text{otherwise} \end{cases}$
- $\gamma(d) = \{\sigma \mid \forall v \in V.\ \forall n \in \mathbb{Z} \,.\, d(v) = n \rightarrow \sigma(v) = n\}$
- α and γ form an onto Galois connection.

COMPUTING ABSTRACT JOP VERSUS LFP: EXAMPLE

```
i := 0;
while(i < n) do
    i := i + 1;
```



Algorithm for computing the abstract JOP will never terminate

However, due to ACC, LFP computation is guaranteed to terminate

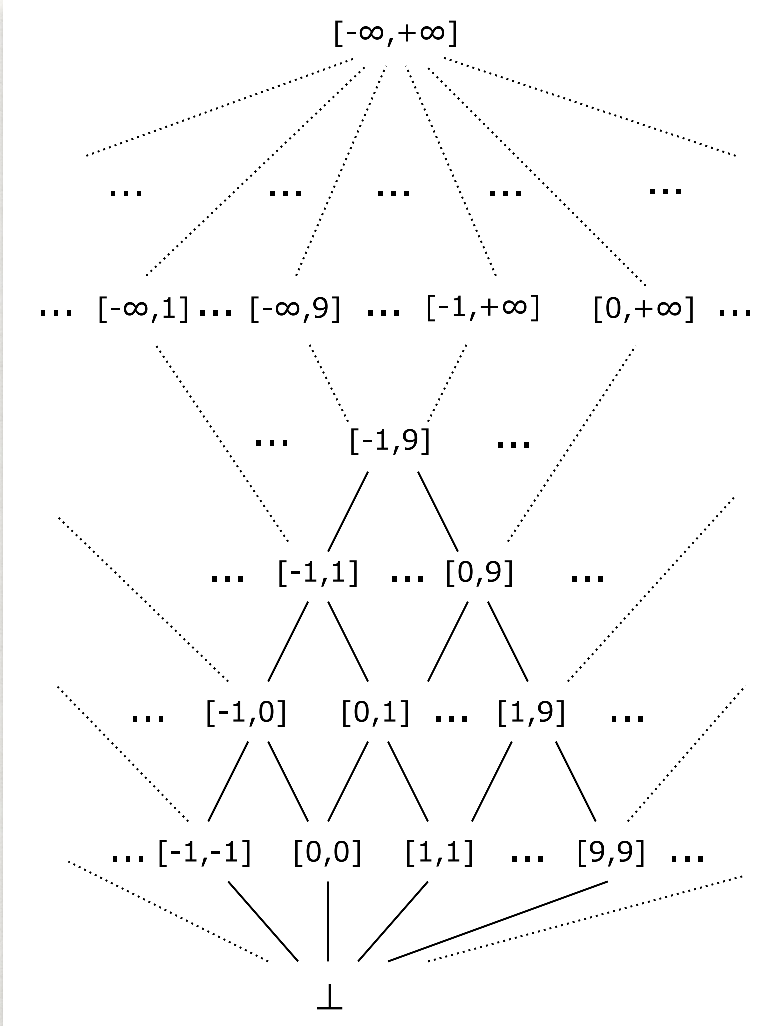
CONSTANT ABSTRACT DOMAIN

ABSTRACT TRANSFER FUNCTIONS

- What will be $\hat{f}_c$ for $c : x := e$?
  - Is $\hat{f}_c$ distributive?
- What will be $\hat{f}_c$ for $c : assume(x = n)$?

INTERVAL ABSTRACT DOMAIN

- $I = \{[a, b] \mid a, b \in \mathbb{Z} \cup \{-\infty, \infty\}\} \cup \{ \perp \}$
  - $D = V \to I$
  - Also called Box abstract domain.
- $[a_1, b_1] \sqsubseteq [a_2, b_2] \Leftrightarrow a_2 \leq a_1 \wedge b_1 \leq b_2$,
  $\forall d \in I \,.\, \perp \sqsubseteq d$
- Is $(I, \sqsubseteq)$ a lattice?
  - $[a_1, b_1] \sqcup [a_2, b_2] = [min(a_1, a_2), max(b_1, b_2)]$
- Is $(I, \sqsubseteq)$ a complete lattice?
  - Maximal element?
- $(D, \sqsubseteq)$:
  $\forall d_1, d_2 \in D \,.\, d_1 \sqsubseteq d_2 \Leftrightarrow \forall v \in V \,.\, d_1(v) \sqsubseteq d_2(v)$



INTERVAL ABSTRACT DOMAIN

ABSTRACTION AND CONCRETIZATION FUNCTION

- $\alpha : \mathbb{P}(States) \rightarrow D, \gamma : D \rightarrow \mathbb{P}(States)$
- $\alpha(c) = d$
  - $d(v) = [min\{\sigma(v) \,|\, \sigma \in c\}, max\{\sigma(v) \,|\, \sigma \in c\}]$
- $\gamma(d) = \{\sigma \mid \forall v \in V \,.\, d(v) = [a, b] \Rightarrow a \leq \sigma(v) \leq b\}$
- Is $(\mathbb{P}(States), \subseteq) \underset{\gamma}{\overset{\alpha}{\rightleftarrows}} (D, \sqsubseteq)$ a Galois Connection?
  - Is it an Onto Galois Connection?

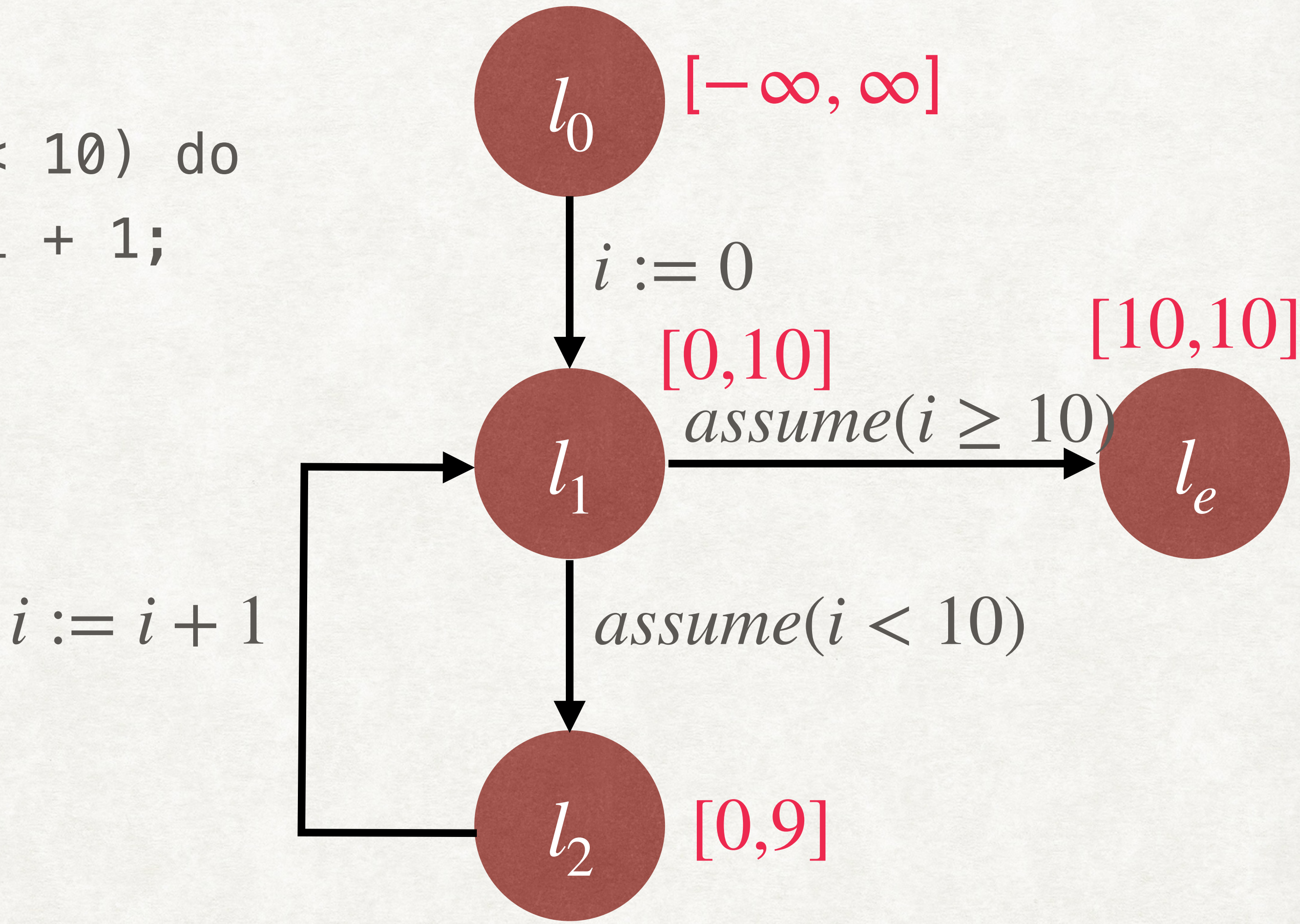
INTERVAL ABSTRACT DOMAIN

ABSTRACT TRANSFER FUNCTION

- Consider $c : x := x + y$
  - We can use interval arithmetic for $\hat{f}_c$
- Assuming $d(x) = [l_x, u_x], d(y) = [l_y, u_y]$
  - $\hat{f}_c(d) = d[x \mapsto [l_x + l_y, u_x + u_y]]$
- Is $\hat{f}_c$ distributive?
  - What about $\hat{f}_c$ for $c : x := x - y$? Is this function distributive?

USING INTERVAL DOMAIN

```
i := 0;
while(i < 10) do
    i := i + 1;
```



USING INTERVAL DOMAIN

```
i := 0;
while(i < n) do
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```

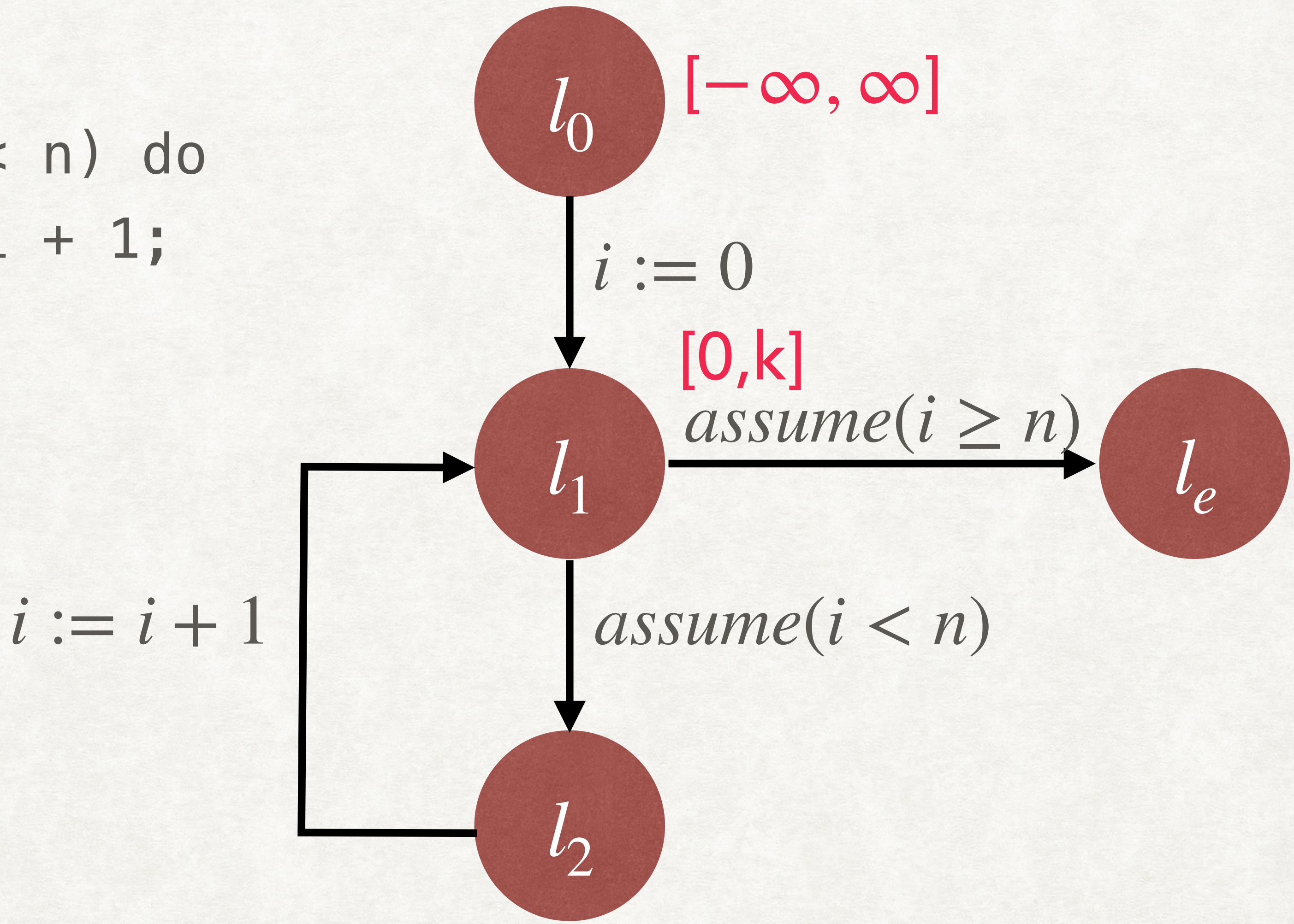

Interval Abstract Domain does not satisfy ACC, hence Kildall's Algorithm may not terminate

WIDENING

- A widening function $\nabla : D \times D \to D$ on a poset $(D, \leq)$ satisfies the following properties:
  - $\forall x, y \in D \,.\, x \sqcup y \leq x \nabla y$
  - For an ascending chain $x_0, x_1, \ldots$, the ascending chain $y_0, y_1, \ldots$ where $y_0 = x_0$ and $y_n = y_{n-1} \nabla x_n$ eventually stabilizes.

WIDENING FOR THE INTERVAL DOMAIN

- We can define the widening operator for interval domain as follows:
  - $[a, b] \triangledown \perp = [a, b]$
  - $\perp \triangledown [a, b] = [a, b]$
  - $[a_1, b_1] \triangledown [a_2, b_2] = [(a_2 < a_1)? -\infty : a_1, (b_1 < b_2)? \infty : b_1]$
- Examples
  - $[1,2] \triangledown [0,2] = ???$

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- Examples
  - $[1,2] \triangledown [0,2] = [-\infty,2]$
  - $[0,2] \triangledown [1,2] = [0,2]$
  - $[2,3] \triangledown [4,6] = ???$

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  - $[a_1, b_1] \triangledown [a_2, b_2] = [(a_2 < a_1)? -\infty : a_1, (b_1 < b_2)? \infty : b_1]$
- Examples
  - $[1,2] \triangledown [0,2] = [-\infty,2]$
  - $[0,2] \triangledown [1,2] = [0,2]$
  - $[2,3] \triangledown [4,6] = [2,\infty]$

KILDALL'S ALGORITHM WITH WIDENING

AbstractForwardPropagate(Γ_c,P)

  S := $\{l_0\}$;

  $\hat{\mu}_K(l_0)$:= α(P);

  $\hat{\mu}_K(l)$:= $\bot$, for $l \in L \backslash \{l_0\}$;

  while S $\neq \varnothing$ do{

    l := Choose S;

    S := S \ $\{l\}$;

    foreach $(l, c, l') \in T$ do{

      F := $\hat{f}_c(\hat{\mu}_K(l))$;

      if $\neg(\mathsf{F} \leq \hat{\mu}_K(l'))$ then{

        $\hat{\mu}_K(l') := \hat{\mu}_K(l') \triangledown F;$

        S := S $\cup \{l'\}$;

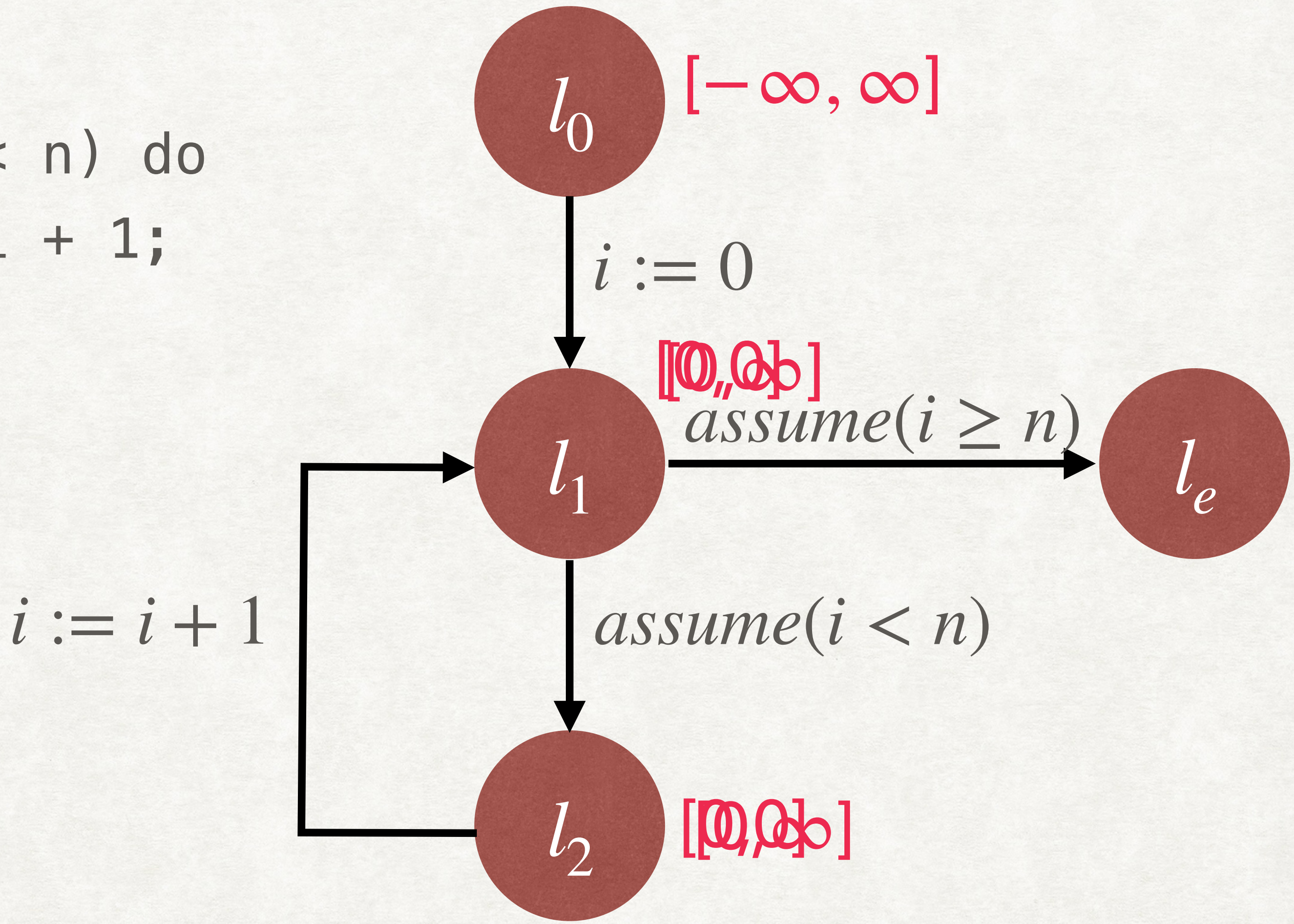
      }

    }

  }

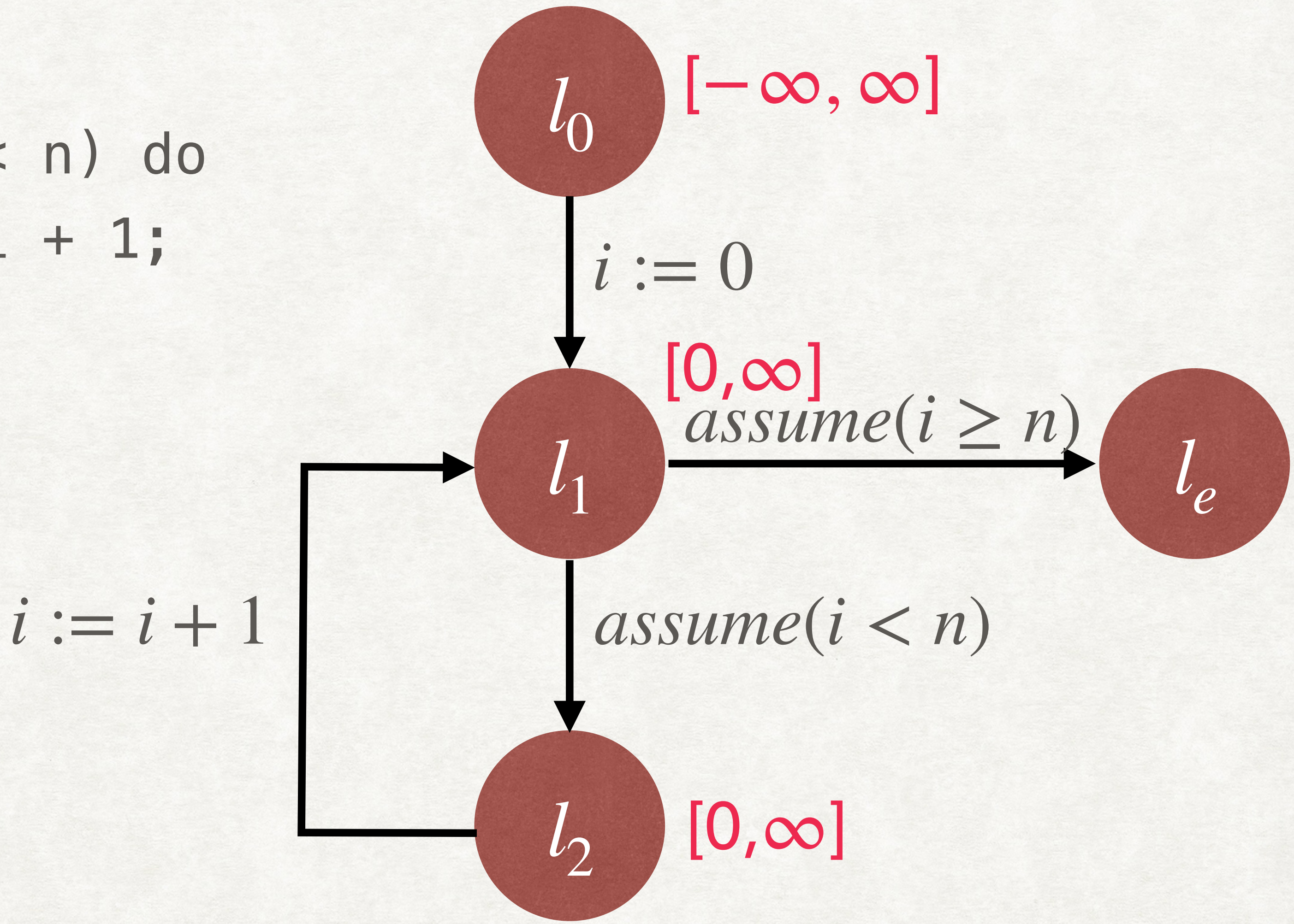
WIDENING EXAMPLE

```
i := 0;
while(i < n) do
    i := i + 1;
```

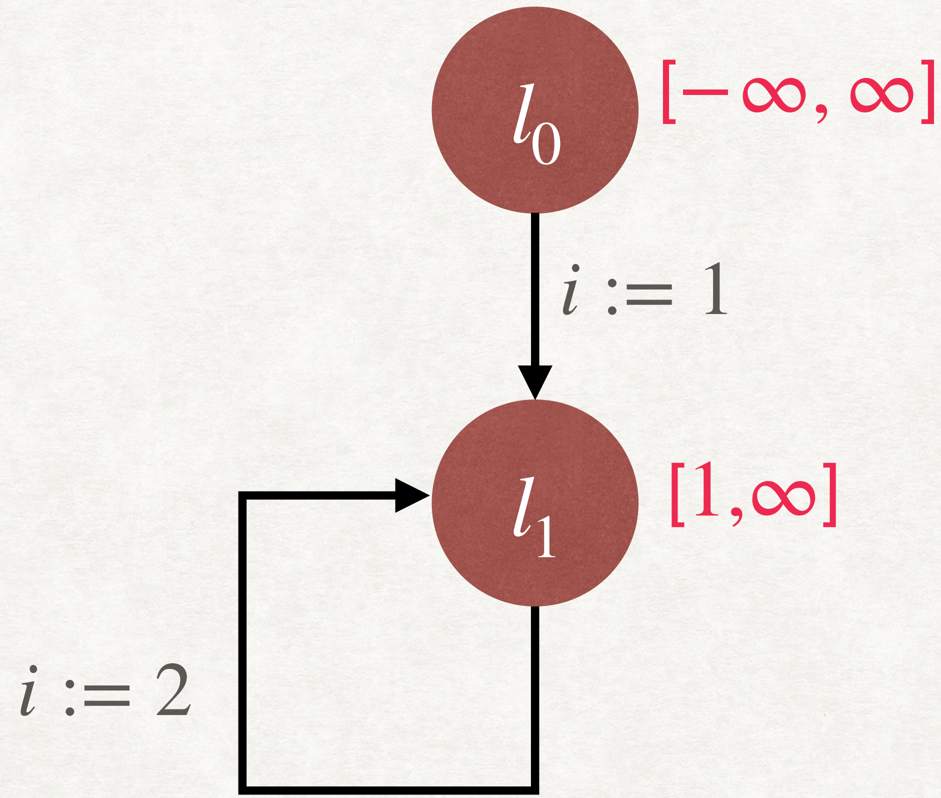


WIDENING EXAMPLE

```
i := 0;
while(i < n) do
    i := i + 1;
```



ANOTHER WIDENING EXAMPLE



NARROWING

- A narrowing function $\triangle : D \times D \to D$ on a poset $(D, \leq)$ satisfies the following properties:
  - $\forall x, y \in D \,.\, y \leq x \Rightarrow y \leq x \triangle y \leq x$
  - For a decreasing chain $x_0 \geq x_1 \geq \ldots$, the decreasing chain $y_0, y_1, \ldots$ where $y_0 = x_0$ and $y_n = y_{n-1} \triangle x_n$ eventually stabilizes.

NARROWING FOR THE INTERVAL DOMAIN

- We can define the narrowing operator for interval domain as follows:
  - $[a, b] \bigtriangleup \bot = \bot$
  - $[a_1, b_1] \bigtriangleup [a_2, b_2] = [(a_1 = -\infty)?a_2 : a_1, (b_1 = \infty)?b_2 : b_1]$
- Examples
  - $[1,3] \bigtriangleup [1,2] =$
  - $[-\infty,6] \bigtriangleup [1,3] =$

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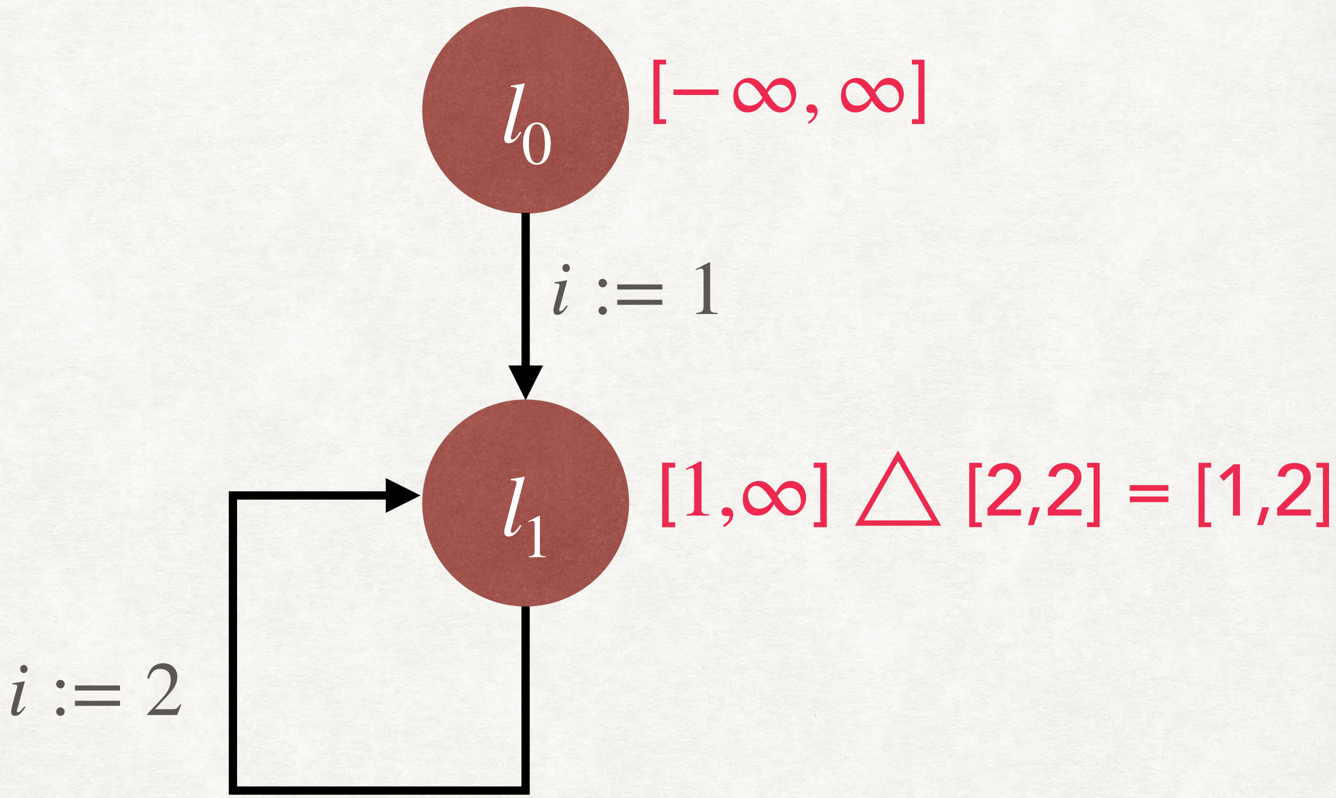
- Examples
  - $[1,3] \bigtriangleup [1,2] = [1,3]$
  - $[-\infty,6] \bigtriangleup [1,3] = [1,6]$

NARROWING EXAMPLE



Apply Narrowing pass after Widening

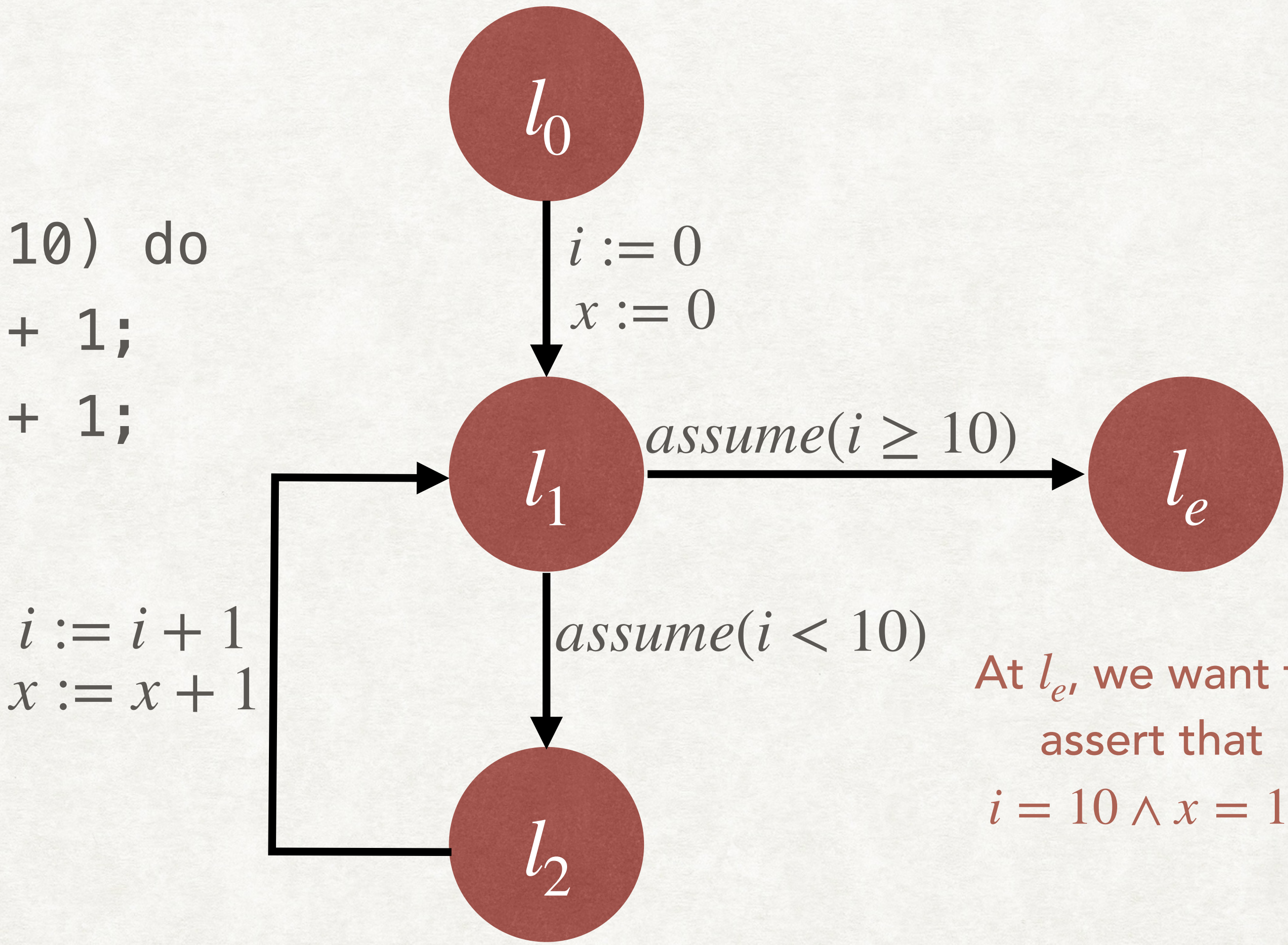
RELATIONAL DOMAINS

- Both the sign and the interval abstract domains are non-relational, i.e. they do not track relationships between variables.
- Relational domains track relationships between variables and are more powerful.
- Examples of relational domains
  - Karr's Domain: Tracks equalities between linear expressions (e.g. $x = 2y + z$). For details, refer to BM Chapter 12.
  - Octagon Domain: Constraints of the form $\pm x \pm y \leq c$
  - Polyhedra Domain: Constraints of the form $c_1x_1 + \ldots c_nx_n \leq c$
- You can experiment with different abstract domains here: `http://pop-art.inrialpes.fr/interproc/interprocweb.cgi`.

THE NEED FOR RELATIONAL DOMAINS

EXAMPLE

```
i := 0;
x := 0;
while(i < 10) do
    i := i + 1;
    x := x + 1;
```



At l_e, we want to assert that $i = 10 \wedge x = 10$

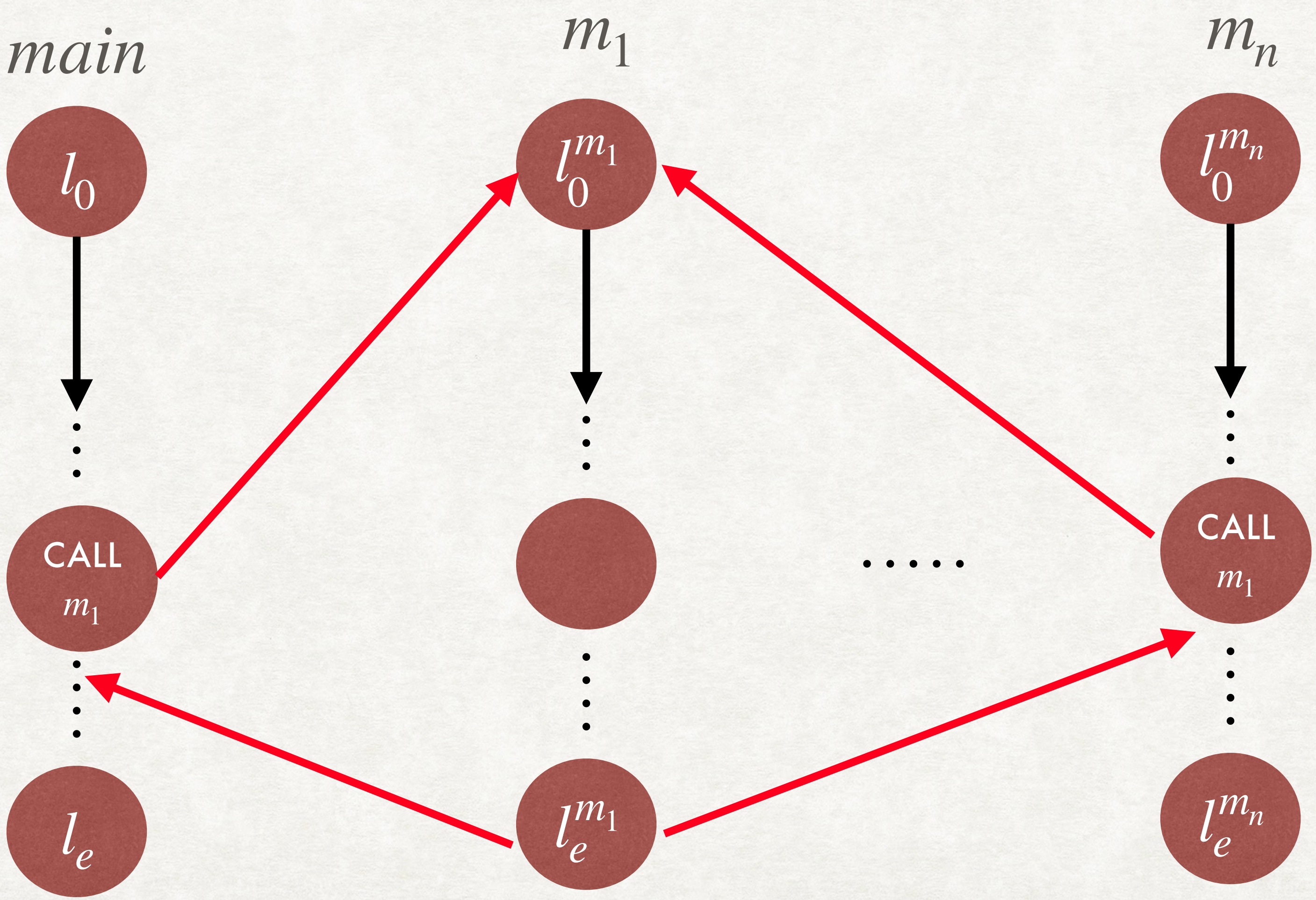
Using the interval domain, we will only be able to show $i = 10$. We need the invariant $i = x$ to show $x = 10$.

INTER-PROCEDURAL ABSTRACT INTERPRETATION

- For programs with multiple functions, we first consider the inter-procedural LTS:

INTER-PROCEDURAL ABSTRACT INTERPRETATION

- For programs with multiple functions, we first consider the inter-procedural LTS:



INTER-PROCEDURAL ABSTRACT INTERPRETATION

- Assuming that variable names are distinct across functions, function call and return statements can be replaced by assignments to parameters and return variables.
- However, the challenge is to only consider inter-procedurally valid paths.
- Naively applying AI on the inter-procedural LTS will result in highly imprecise analysis.

SHARIR AND PNUELI'S APPROACHES TO INTER-PROCEDURAL AI

- Call-Strings based approach
  - Change the abstract domain to also record the history of call-sites.
  - Since call-strings can be infinite in size, two practical approaches are also proposed: Approximate call-string method and Bounded call-string method.
- Functional approach
  - Maintain an abstract summary of every method which maps abstract value of input parameter(s) to abstract value of return variable.
  - Abstract summaries calculated on-the-fly during the analysis.

Micha Sharir and Amir Pnueli: Two approaches to interprocedural data flow analysis (1981)

LIMITATIONS OF ABSTRACT INTERPRETATION

- Precision depends upon the choice of the abstract domain.
- Hard to choose the right abstract domain: may depend on the program and the specification.
- Hard to interpret a negative result
  - If verification fails, then we don’t know whether the program is actually incorrect, or the abstract domain was not precise enough.
  - No counterexample is provided as output.