FIRST-ORDER LOGIC

SYNTAX

Term	Constants - a,b,c… Variables - x,y,z…
Function	Arity *n*: Takes *n* terms as input, and forms a term
Predicate	Arity *n*: Takes *n* terms as input, and forms an atom

SYNTAX

Atom	Predicate: p,q,r…
Logical Connectives	$\wedge$: and, $\vee$: or, $\neg$: not, $\rightarrow$: implies, $\leftrightarrow$: if and only if(iff)
Quantifier	$\forall$: Universal $\exists$: Existential
Literal	Atom or its negation
Formula	A literal or the application of logical connectives and quantifiers to formulae

EXAMPLE

$$\forall x \,.\, ((\exists y \,.\, p(f(x), y)) \rightarrow q(x))$$

EXAMPLE

$$\forall x\,.\,((\exists y\,.\,p(f(x), y)) \rightarrow q(x))$$

Variables

EXAMPLE

$$\forall x \,.\, ((\exists y \,.\, p(f(x), y)) \rightarrow q(x))$$

Function

EXAMPLE

$$\forall x \,.\, ((\exists y \,.\, p(f(x), y)) \rightarrow q(x))$$

Predicate

EXAMPLE

$$\forall x\,.\,((\exists y\,.\,p(f(x), y)) \rightarrow q(x))$$

Quantifier

EXAMPLE

$$\forall x \,.\, ((\exists y \,.\, p(f(x), y)) \rightarrow q(x))$$

Scope of x

EXAMPLE

$$\forall x \,.\, ((\exists y \,.\, p(f(x), y)) \rightarrow q(x))$$

Scope of y

EXAMPLE

$$\forall x \,.\, ((\exists y \,.\, p(f(x), y)) \rightarrow q(x))$$

Scope of y

An occurrence of a variable is bound if it is in the scope of some quantifier

EXAMPLE

$$\forall x \,.\, ((\exists y \,.\, p(f(x), y)) \rightarrow q(x))$$

Scope of y

An occurrence of a variable is bound if it is in the scope of some quantifier

An occurrence of a variable is free if it is not in the scope of some quantifier

SEMANTICS - EXAMPLES

- All Humans are mortal.
  - Assume unary predicates *human* and *mortal.*

$$\forall x \,.\, human(x) \rightarrow mortal(x)$$

SEMANTICS - EXAMPLES

- There always exists someone such that if (s)he laughs, then everyone laughs.
    - Assume unary predicate *laughs*.

$$\exists x \,.\, (laughs(x) \rightarrow \forall y \,.\, laughs(y))$$

SEMANTICS - EXAMPLES

- Every dog has its day.
  - $\forall x \,.\, dog(x) \rightarrow \exists y \,.\, day(y) \wedge itsDay(x, y)$
- Some dogs have more days than others.
  - $\exists x, y \,.\, dog(x) \wedge dog(y) \wedge \#days(x) > \#days(y)$
- All cats have more days than dogs.
  - $\forall x, y \,.\, (dog(x) \wedge cat(y)) \rightarrow \#days(y) > \#days(x)$

INTERPRETATIONS

- An interpretation I is an assignment from variables (in general, terms) to values in a specified domain.
- Domain, D_I
  - A nonempty set of values or objects. Also called universe of discourse.
  - Numbers, humans, students, courses, animals,…
- Assignment, α_I
  - Maps constants and variables to elements of the domain D_I (i.e. values)
  - Maps functions and predicate symbols to functions and predicates (of the same arity) over D_I

INTERPRETATIONS - EXAMPLE 1

- Suppose $D_I = \{A, B\}$
- Constants a and b are mapped to following elements in D_I
  - $\alpha_I(a) = B \quad \alpha_I(b) = A$
- A binary function symbol f is mapped to the following actual function on D_I:
  - $\alpha_I(f) = \{(A, A) \to B,\ (A, B) \to B,\ (B, A) \to A,\ (B, B) \to B\}$
- A unary predicate symbol p is mapped to the following actual predicate on D_I
  - $\alpha_I(p) = \{A \to True,\ B \to False\}$

INTERPRETATIONS - EXAMPLE 2

- Consider the formula: $x + y > z \rightarrow y > z - x$
  - Here, $+, -$ are functions and $>$ is a predicate.
  - Equivalent to $p(f(x, y), z) \rightarrow p(y, g(z, x))$.
- A standard interpretation for this formula would be:
  - Domain: $\mathbb{Z}$
  - $+, -$ would be mapped to the standard integer addition and subtraction functions.
  - $>$ would be mapped to the standard greater-than relation over integers.
  - x, y, z could be mapped to $5,10,9$ resp.

SEMANTICS: INDUCTIVE DEFINITION

Base Case:

$I \vDash \top$	
$I \nvDash \bot$	
$I \vDash p$	iff I[p]=true
$I \nvDash p$	iff I[p]=false

Inductive Case:

$I \vDash \neg F$	iff $I \nvDash F$
$I \vDash F_1 \wedge F_2$	iff $I \vDash F_1$ and $I \vDash F_2$
$I \vDash F_1 \vee F_2$	iff $I \vDash F_1$ or $I \vDash F_2$
$I \vDash F_1 \rightarrow F_2$	iff $I \nvDash F_1$ or $I \vDash F_2$
$I \vDash F_1 \leftrightarrow F_2$	iff $I \vDash F_1$ and $I \vDash F_2$, or $I \nvDash F_1$ and $I \nvDash F_2$

SEMANTICS: INDUCTIVE DEFINITION

Base Case:

What does this mean?

$I \vDash \top$	
$I \nvDash \bot$	
$I \vDash p$	iff I[p]=true
$I \nvDash p$	iff I[p]=false

Inductive Case:

$I \vDash \neg F$	iff $I \nvDash F$
$I \vDash F_1 \wedge F_2$	iff $I \vDash F_1$ and $I \vDash F_2$
$I \vDash F_1 \vee F_2$	iff $I \vDash F_1$ or $I \vDash F_2$
$I \vDash F_1 \rightarrow F_2$	iff $I \nvDash F_1$ or $I \vDash F_2$
$I \vDash F_1 \leftrightarrow F_2$	iff $I \vDash F_1$ and $I \vDash F_2$, or $I \nvDash F_1$ and $I \nvDash F_2$

SEMANTICS - CONTINUED…

$$I \models p(t_1, \ldots, t_n) \text{ iff } \alpha_I[p](\alpha_I[t_1], \ldots, \alpha_I[t_n]) = \top$$

$$\alpha_I[f(t_1, \ldots, t_n)] = \alpha_I[f](\alpha_I[t_1], \ldots, \alpha_I[t_n])$$

SEMANTICS - EXAMPLE

INTERPRETATION I

$D_I = \{A, B\}$

$\alpha_I(a) = B \quad \alpha_I(b) = A$

$\alpha_I(f) = \{(A, A) \rightarrow B,\ (A, B) \rightarrow B,\ (B, A) \rightarrow A,\ (B, B) \rightarrow B\}$

$\alpha_I(p) = \{A \rightarrow True,\ B \rightarrow False\}$

$$I \models p(b)$$

$$I \models p(f(a, b))$$

$$I \not\models p(f(b, a))$$

SEMANTICS - QUANTIFIERS

- An $x-$variant of interpretation $I = (D_I, \alpha_I)$ is an interpretation $J = (D_J, \alpha_J)$ such that
  - $D_I = D_J$;
  - and $\alpha_I[y] = \alpha_J[y]$ for all constant, free variable, function, and predicate symbols y, except possibly x.
- An $x-$variant of I, where x is mapped to some $v \in D_I$ is denoted by $I[x \mapsto v]$.

$$I \vDash \forall x\,.\,F \text{ iff for all } v \in D_I, I[x \mapsto v] \vDash F$$

$$I \vDash \exists x\,.\,F \text{ iff there exists } v \in D_I, I[x \mapsto v] \vDash F$$

SEMANTICS - QUANTIFIERS - EXAMPLE

INTERPRETATION I

$D_I = \{A, B\}$

$\alpha_I(a) = B \quad \alpha_I(b) = A$

$\alpha_I(f) = \{(A, A) \to B,\ (A, B) \to B,\ (B, A) \to A,\ (B, B) \to B\}$

$\alpha_I(p) = \{A \to True,\ B \to False\}$

$$I \vDash \exists x\,.\,p(x)$$

$$I \vDash \forall x\,.\,\neg p(f(b, x))$$

SATISFIABILITY AND VALIDITY

- A FOL formula F is satisfiable if there exists an interpretation I such that $I \vDash F$.
  - If no such interpretation exists, then it is unsatisfiable
- A FOL formula F is valid if for all interpretations I, $I \vDash F$
  - Technically, only for interpretations which assign to all the constants, variables, predicates, functions used in F.
- F is valid iff $\neg F$ is unsatisfiable.

FREE VARIABLES

- Given a FOL formula F, a variable x is free in F if there is a use of x in F which is not bound to any quantifier.
  - $free(F)$ denotes all variables free in F.
- A FOL formula F is closed if it does not contain any free variables.
- Technically, satisfiability and validity are only applicable for closed FOL formulae.
- However, we can extend these concepts to formulae with free variables by following the below convention:
  - For satisfiability, all free variables are implicitly existentially quantified.
  - For validity, all free variables are implicitly universally quantified.

SATISFIABILITY AND VALIDITY

EXAMPLES

- Is the formula $\forall x \,.\, \exists y \,.\, p(x, y)$ satisfiable?
  - Yes. A satisfying interpretation:
    $I = (\{A\}, < p \mapsto \{(A, A) \mapsto \top \} >)$
- Is the formula $\forall x \,.\, \exists y \,.\, p(x, y)$ valid?
  - No. A falsifying interpretation:
    $I = (\{A\}, < p \mapsto \{(A, A) \mapsto \bot \} >)$
- Is the formula $(\forall x \,.\, p(x)) \rightarrow (\exists y \,.\, p(y))$ valid?
- Is the formula $\forall x \,.\, (p(x) \rightarrow (\exists y \,.\, p(y)))$ valid?
  - What about $\forall x \,.\, (p(x) \rightarrow (\forall y \,.\, p(y)))$?

DECISION PROCEDURE FOR VALIDITY

- Semantic Argument Method
  - Deductive Approach
  - Proof by Contradiction
  - Assume that a falsifying interpretation exists.
  - Use proof rules to deduce more facts.
  - The goal is to find contradictory facts in each branch (also called closing the branch).
- Proof rules for negation, conjunction, disjunction, implication, iff carry over from Propositional logic

PROOF RULES

UNIVERSAL QUANTIFICATION

$$\frac{I \models \forall x \,.\, F}{I[x \mapsto v] \models F} \quad \text{For any } v \in D_I$$

PROOF RULES

UNIVERSAL QUANTIFICATION

$$\frac{I \models \forall x\,.\,F}{I[x \mapsto v] \models F}$$ For any $v \in D_I$

$$\frac{I \not\models \forall x\,.\,F}{I[x \mapsto v] \not\models F}$$ For a fresh $v \in D_I$

PROOF RULES

EXISTENTIAL QUANTIFICATION

$$\frac{I \models \exists x \,.\, F}{I[x \mapsto v] \models F}$$

For a fresh $v \in D_I$

PROOF RULES

EXISTENTIAL QUANTIFICATION

$$\frac{I \models \exists x \,.\, F}{I[x \mapsto v] \models F} \quad \text{For a fresh } v \in D_I$$

$$\frac{I \not\models \exists x \,.\, F}{I[x \mapsto v] \not\models F} \quad \text{For any } v \in D_I$$

PROOF RULES

CONTRADICTION

$$\frac{J \models p(s_1, \ldots, s_n) \quad K \not\models p(t_1, \ldots, t_n) \qquad J = I[\ldots] \quad K = I[\ldots] \qquad \alpha_J[s_i] = \alpha_K[t_i] \text{ for all } i = 1, \ldots, n}{I \models \bot}$$

EXAMPLE

Prove that $(\forall x \,.\, p(x)) \rightarrow (\forall y \,.\, p(y))$ is valid

$$\frac{I \not\models (\forall x \,.\, p(x)) \rightarrow (\forall y \,.\, p(y))}{I \models (\forall x \,.\, p(x)) \qquad I \not\models (\forall y \,.\, p(y))}$$

EXAMPLE

Prove that $(\forall x \,.\, p(x)) \rightarrow (\forall y \,.\, p(y))$ is valid

$$\dfrac{I \not\models (\forall x \,.\, p(x)) \rightarrow (\forall y \,.\, p(y))}{I \models (\forall x \,.\, p(x)) \qquad \dfrac{I \not\models (\forall y \,.\, p(y))}{I[y \mapsto v] \not\models p(y)}\ \text{[for a fresh } v\text{]}}$$

EXAMPLE

Prove that $(\forall x\,.\,p(x)) \rightarrow (\forall y\,.\,p(y))$ is valid

$$\dfrac{I \not\models (\forall x\,.\,p(x)) \rightarrow (\forall y\,.\,p(y))}{\dfrac{I \models (\forall x\,.\,p(x))}{I[x \mapsto v] \models p(x)} \qquad \dfrac{I \not\models (\forall y\,.\,p(y))}{I[y \mapsto v] \not\models p(y)}}$$

[for a fresh v]

EXAMPLE

Prove that $(\forall x \,.\, p(x)) \rightarrow (\forall y \,.\, p(y))$ is valid

$$\dfrac{I \not\models (\forall x \,.\, p(x)) \rightarrow (\forall y \,.\, p(y))}{\dfrac{I \models (\forall x \,.\, p(x))}{I[x \mapsto v] \models p(x)} \qquad \dfrac{I \not\models (\forall y \,.\, p(y))}{I[y \mapsto v] \not\models p(y)}}$$

[for a fresh v]

CONTRADICTION

EXAMPLE - 2

Prove that $\exists x \,.\, F \to G \leftrightarrow (\forall x \,.\, F) \to (\exists x \,.\, G)$ is valid

EXAMPLE - 2

Prove that $\exists x \,.\, F \rightarrow G \leftrightarrow (\forall x \,.\, F) \rightarrow (\exists x \,.\, G)$ is valid

$$I \not\models \exists x \,.\, F \rightarrow G \leftrightarrow (\forall x \,.\, F) \rightarrow (\exists x \,.\, G)$$

EXAMPLE - 2

Prove that $\exists x\,.\,F \rightarrow G \leftrightarrow (\forall x\,.\,F) \rightarrow (\exists x\,.\,G)$ is valid

$$I \not\models \exists x\,.\,F \rightarrow G \leftrightarrow (\forall x\,.\,F) \rightarrow (\exists x\,.\,G)$$

$I \models \exists x\,.\,F \rightarrow G \quad I \not\models (\forall x\,.\,F) \rightarrow (\exists x\,.\,G)$ | $I \not\models \exists x\,.\,F \rightarrow G \quad I \models (\forall x\,.\,F) \rightarrow (\exists x\,.\,G)$

EXAMPLE - 2

Prove that $\exists x . F \rightarrow G \leftrightarrow (\forall x . F) \rightarrow (\exists x . G)$ is valid

$$I \not\models \exists x . F \rightarrow G \leftrightarrow (\forall x . F) \rightarrow (\exists x . G)$$

$I \models \exists x . F \rightarrow G$ $\qquad$ $I \not\models (\forall x . F) \rightarrow (\exists x . G)$

EXAMPLE - 2

Prove that $\exists x . F \rightarrow G \leftrightarrow (\forall x . F) \rightarrow (\exists x . G)$ is valid

$$I \not\models \exists x . F \rightarrow G \leftrightarrow (\forall x . F) \rightarrow (\exists x . G)$$

$$I \models \exists x . F \rightarrow G \qquad I \not\models (\forall x . F) \rightarrow (\exists x . G)$$

$$I[x \mapsto v] \models F \rightarrow G$$

EXAMPLE - 2

Prove that $\exists x\,.\,F \rightarrow G \leftrightarrow (\forall x\,.\,F) \rightarrow (\exists x\,.\,G)$ is valid

$$\dfrac{I \not\models \exists x\,.\,F \rightarrow G \leftrightarrow (\forall x\,.\,F) \rightarrow (\exists x\,.\,G)}{\dfrac{\dfrac{I \models \exists x\,.\,F \rightarrow G}{I[x \mapsto v] \models F \rightarrow G}}{I[x \mapsto v] \not\models F \quad \mid \quad I[x \mapsto v] \models G} \qquad I \not\models (\forall x\,.\,F) \rightarrow (\exists x\,.\,G)}$$

EXAMPLE - 2

Prove that $\exists x . F \rightarrow G \leftrightarrow (\forall x . F) \rightarrow (\exists x . G)$ is valid

$$I \not\models \exists x . F \rightarrow G \leftrightarrow (\forall x . F) \rightarrow (\exists x . G)$$

$$I \models \exists x . F \rightarrow G \qquad I \not\models (\forall x . F) \rightarrow (\exists x . G)$$

$$I[x \mapsto v] \models F \rightarrow G \qquad I \models (\forall x . F) \quad I \not\models (\exists x . G)$$

$$I[x \mapsto v] \not\models F \quad | \quad I[x \mapsto v] \models G$$

EXAMPLE - 2

Prove that $\exists x \,.\, F \rightarrow G \leftrightarrow (\forall x \,.\, F) \rightarrow (\exists x \,.\, G)$ is valid

$$I \not\models \exists x \,.\, F \rightarrow G \leftrightarrow (\forall x \,.\, F) \rightarrow (\exists x \,.\, G)$$

$$I \models \exists x \,.\, F \rightarrow G \qquad I \not\models (\forall x \,.\, F) \rightarrow (\exists x \,.\, G)$$

$$I[x \mapsto v] \models F \rightarrow G \qquad I \models (\forall x \,.\, F) \quad I \not\models (\exists x \,.\, G)$$

$$I[x \mapsto v] \not\models F \quad | \quad I[x \mapsto v] \models G \qquad I[x \mapsto v] \models F$$

EXAMPLE - 2

Prove that $\exists x . F \rightarrow G \leftrightarrow (\forall x . F) \rightarrow (\exists x . G)$ is valid

$$I \not\models \exists x . F \rightarrow G \leftrightarrow (\forall x . F) \rightarrow (\exists x . G)$$

$$I \models \exists x . F \rightarrow G \qquad I \not\models (\forall x . F) \rightarrow (\exists x . G)$$

$$I[x \mapsto v] \models F \rightarrow G \qquad I \models (\forall x . F) \quad I \not\models (\exists x . G)$$

$$I[x \mapsto v] \not\models F \quad | \quad I[x \mapsto v] \models G \qquad I[x \mapsto v] \models F$$

CONTRADICTION

EXAMPLE - 2

Prove that $\exists x\,.\,F \rightarrow G \leftrightarrow (\forall x\,.\,F) \rightarrow (\exists x\,.\,G)$ is valid

$$I \not\models \exists x\,.\,F \rightarrow G \leftrightarrow (\forall x\,.\,F) \rightarrow (\exists x\,.\,G)$$

$$I \models \exists x\,.\,F \rightarrow G \qquad\qquad I \not\models (\forall x\,.\,F) \rightarrow (\exists x\,.\,G) \;\Big|$$

$$I[x \mapsto v] \models F \rightarrow G \qquad\qquad I \models (\forall x\,.\,F) \quad I \not\models (\exists x\,.\,G)$$

$$I[x \mapsto v] \not\models F \;\Big|\; I[x \mapsto v] \models G \quad I[x \mapsto v] \models F \quad I[x \mapsto v] \not\models G$$

EXAMPLE - 2

Prove that $\exists x \,.\, F \rightarrow G \leftrightarrow (\forall x \,.\, F) \rightarrow (\exists x \,.\, G)$ is valid

$$I \not\models \exists x \,.\, F \rightarrow G \leftrightarrow (\forall x \,.\, F) \rightarrow (\exists x \,.\, G)$$

$$I \models \exists x \,.\, F \rightarrow G \qquad\qquad I \not\models (\forall x \,.\, F) \rightarrow (\exists x \,.\, G)$$

$$I[x \mapsto v] \models F \rightarrow G \qquad\qquad I \models (\forall x \,.\, F) \quad I \not\models (\exists x \,.\, G)$$

$$I[x \mapsto v] \not\models F \quad \Big| \quad I[x \mapsto v] \models G \qquad I[x \mapsto v] \models F \quad I[x \mapsto v] \not\models G$$

CONTRADICTION

Homework: Complete the proof in the other branch

MORE EXAMPLES

- Prove or disprove validity of following FOL formulae
  - $\forall x \,.\, F \rightarrow G \leftrightarrow (\exists x \,.\, F) \rightarrow (\forall x \,.\, G)$
  - $(\forall x \,.\, p(x)) \leftrightarrow \neg(\exists x \,.\, \neg p(x))$
  - $(\exists x \,.\, p(x)) \rightarrow (\forall y \,.\, p(y))$
  - $\exists x \,.\, (p(x) \rightarrow \forall y \,.\, p(y))$

DECIDABILITY OF VALIDITY OF FOL

- Church and Turing showed that it is undecidable to find whether a first-order formula is valid or not.
- But we have just seen the Semantic Argument-based decision procedure!
  - How to instantiate domain values in Proof rules for quantifiers?
  - What order should proof rules be applied in?
- The semantic argument-based method can be augmented to make the validity of FOL problem semi-decidable.
  - If the input formula is valid, then the method will halt and answer positive.
  - If the input formula is not valid, then the method may never halt.
  - More details in the BM book [Chapter 2, Section 2.7].

NORMAL FORMS OF FOL

- Negation Normal Form (NNF)
  - Should use only $\neg$, $\wedge$, $\vee$ as the logical connectives, and $\neg$ should only be applied to literals
  - $\neg(\forall x\,.\,F) \Leftrightarrow \exists x\,.\,\neg F$ and $\neg(\exists x\,.\,F) \Leftrightarrow \forall x\,.\,\neg F$

PRENEX NORMAL FORM

- A formula is in Prenex Normal Form (PNF) if all of its quantifiers appear at the beginning of the formula:
  - $Q_1 x_1 \ldots Q_n x_n . F[x_1, \ldots, x_n]$, where F is quantifier-free and may have $x_1, \ldots, x_n$ as free variables.
- How to convert an arbitrary formula F to PNF?

1. First, convert F to NNF (call it F_1).
2. If two quantified variables in F_1 have the same name, then rename them to fresh variables (obtaining the formula F_2).
3. Remove all quantifiers in F_2 to obtain F_3.
4. Add all the removed quantifiers at the beginning of F_3, ensuring that if Q_j was in the scope of Q_i in F_2, then Q_i occurs before Q_j

PRENEX NORMAL FORM

EXAMPLE

$$F:\ \forall x.\ \neg(\exists y.\ p(x,y)\ \wedge\ p(x,z))\ \vee\ \exists y.\ p(x,y)$$

STEP-1

$$F_1:\ \forall x.\ (\forall y.\ \neg p(x,y)\ \vee\ \neg p(x,z))\ \vee\ \exists y.\ p(x,y)$$

STEP-2

$$F_2:\ \forall x.\ (\forall y.\ \neg p(x,y)\ \vee\ \neg p(x,z))\ \vee\ \exists w.\ p(x,w)$$

STEP-3

$$F_3:\ \neg p(x,y)\ \vee\ \neg p(x,z)\ \vee\ p(x,w)$$

STEP-4

$$\forall x.\ \forall y.\ \exists w.\ \neg p(x,y)\ \vee\ \neg p(x,z)\ \vee\ p(x,w)$$

QUESTIONS

- Semantic Argument-based method for Validity of FOL Formula
  - Can it be directly used for checking satisfiability of FOL Formula?
  - No. Consider the formula $\forall x . \exists y . p(x, y)$.
    - Applying the proof rules will result in an interpretation which is not satisfying.
- Why do we insist on fresh values in some proof rules?
  - Example: $\exists x . \exists y . \exists z . p(x, y) \vee \neg p(x, z)$.
- Prenex Normal Form
  - Is $\forall x . \exists y . p(x, y) \Leftrightarrow \exists y . \forall x . p(x, y)$?
  - What about $\forall x . \exists y . p(x) \wedge q(y)$ and $\exists y . \forall x . p(x) \wedge q(y)$?

HOMEWORK

- Please try Exercises 2.1-2.4 in the BM Book, Chapter 2.