HOARE LOGIC

HOARE LOGIC

INTRODUCTION

- Since finding the exact *wp* or *sp* for while-loops is difficult, we will use an over-approximation in the form of an inductive invariant which preserves soundness.
  - Much of the rest of the course (and majority of research in verification) deals with how to handle the verification problem for loops/loop-like constructs.
- Hoare Logic is a program logic/verification strategy which can be directly used to prove the validity of Hoare Triples.
  - Also provides a framework for specifying and verifying Inductive Loop Invariants.

DEFINITION

- Given sets of states P and Q, a program c satisfies the specification $\{P\}\mathsf{c}\{Q\}$ if:
  - $\forall \sigma, \sigma' . \sigma \in P \wedge (\sigma, \mathsf{c}) \hookrightarrow^* (\sigma', \mathsf{skip}) \Rightarrow \sigma' \in Q$
- Using FOL formulae P and Q to express sets of states, we can now use the symbolic semantics $\rho(\mathsf{c})$:
  - $\forall V, V' . P \wedge \rho(\mathsf{c}) \rightarrow Q[V'/V]$
- Hoare Logic is a program logic/proof system to directly prove the validity of Hoare Triples.
- We will study it in two forms:
  - A set of inference rules
  - A procedure to generate verification conditions (VCs) in FOL

RELATION WITH WP AND SP

- How are Hoare Triples, Weakest Pre-condition and Strongest Post-condition related with each other?
  - $\{wp(P, \mathsf{c})\}$ c $\{P\}$
  - $\{P\}$ c $\{sp(P, \mathsf{c})\}$
- Homework: Prove this formally using the definitions!

INFERENCE RULES

FORMAT

PREMISES $\quad \{P_1\}S_1\{Q_1\} \quad \{P_2\}S_2\{Q_2\} \quad \ldots \quad \{P_n\}S_n\{Q_n\}$

CONCLUSION $\quad \{P\}S\{Q\}$

$$\frac{\{P_1\}S_1\{Q_1\} \quad \{P_2\}S_2\{Q_2\} \quad \ldots \quad \{P_n\}S_n\{Q_n\}}{\{P\}S\{Q\}}$$

Key Idea: Use the validity of Hoare triples for smaller statements to establish validity for compound statements

INFERENCE RULES

PRIMITIVE STATEMENTS

$$\frac{}{\{P[\mathsf{e}/\mathsf{x}]\}\ \mathsf{x} := \mathsf{e}\ \{P\}}\ \text{[R-ASSIGN]}$$

$$\frac{}{\{\forall \mathsf{x} \,.\, P\}\ \mathsf{x} := \mathsf{havoc}\ \{P\}}\ \text{[R-HAVOC]}$$

$$\frac{}{\{Q \rightarrow P\}\ \mathsf{assume}(Q)\ \{P\}}\ \text{[R-ASSUME]}$$

$$\frac{}{\{Q \wedge P\}\ \mathsf{assert}(Q)\ \{P\}}\ \text{[R-ASSERT]}$$

EXAMPLES

- Which of the following are true?
  - $\{y = 10\}$ x := 10 $\{y = x\}$
  - $\{x = n - 1\}$ x := x + 1 $\{x = n\}$
  - $\{y = x\}$ y := 2 $\{y = x\}$
  - $\{z = 10\}$ y := 2 $\{z = 10\}$
  - $\{y = 10\}$ y := x $\{y = x\}$
- The last Hoare triple is valid, but we cannot prove it using [R-ASSIGN].
  - According to [R-ASSIGN], we have $\{y = x[x/y]\}$ y := x $\{y = x\}$. Hence, $\{x = x\}$ y := x $\{y = x\}$, which simplifies to $\{\top\}$ y := x $\{y = x\}$.
  - Notice that $y = 10 \Rightarrow \top$.

PRE-CONDITION STRENGTHENING

$$\frac{\{P'\}\ \mathsf{c}\ \{Q\} \qquad P \Rightarrow P'}{\{P\}\ \mathsf{c}\ \{Q\}}\ \text{[R-STRENGTHEN-PRE]}$$

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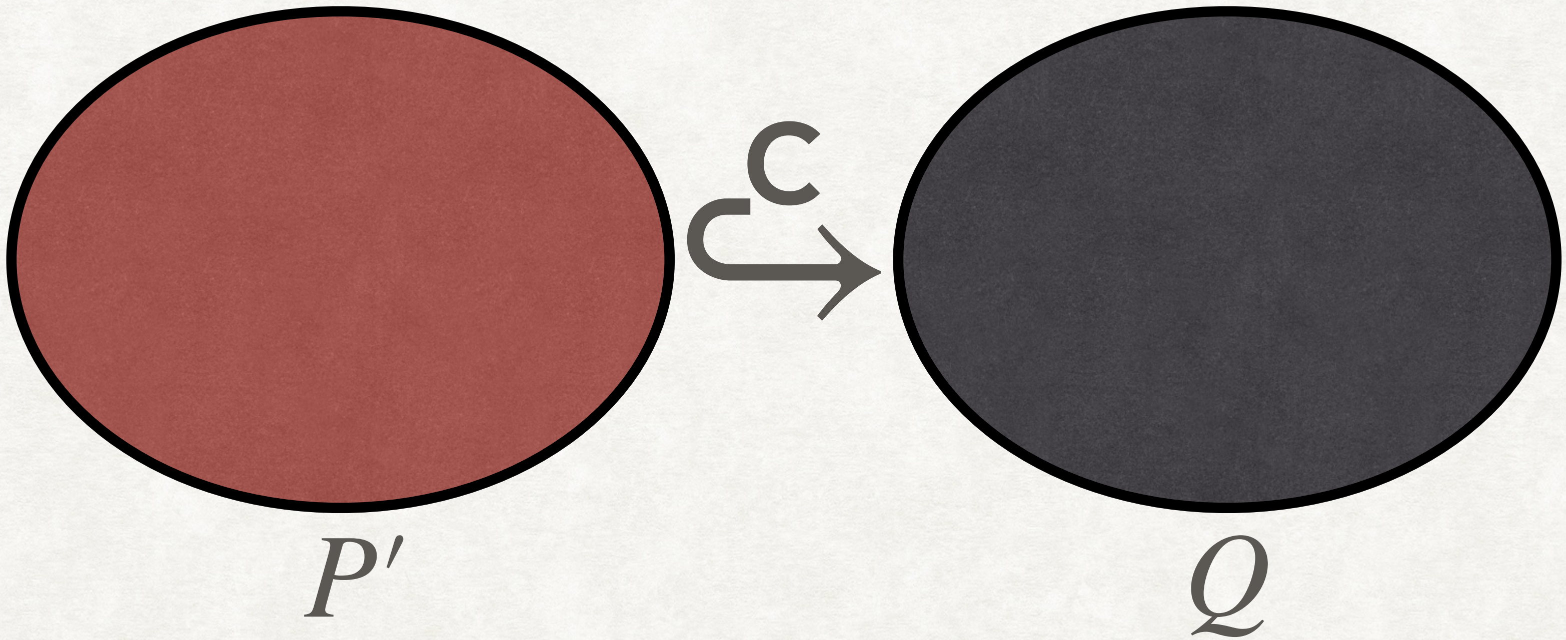
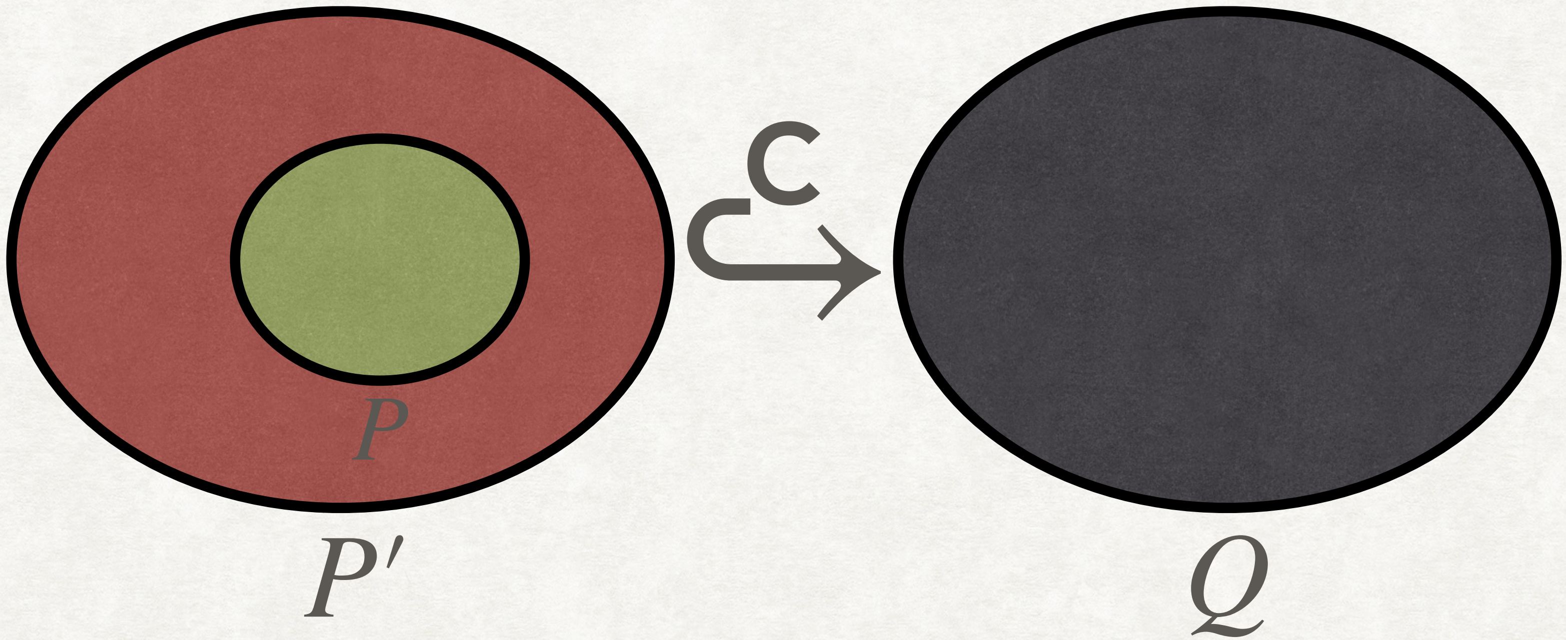


PRE-CONDITION STRENGTHENING

$$\frac{\{P'\}\ \mathsf{c}\ \{Q\} \qquad P \Rightarrow P'}{\{P\}\ \mathsf{c}\ \{Q\}}\ \text{[R-STRENGTHEN-PRE]}$$



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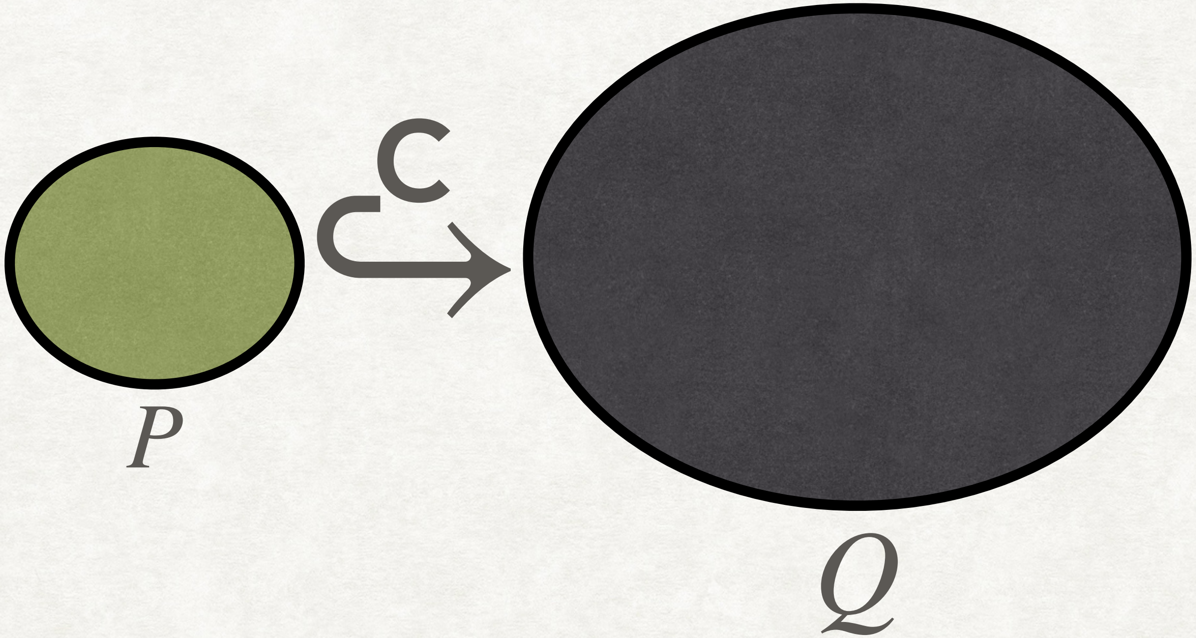

PRE-CONDITION STRENGTHENING

$$\frac{\{P'\}\ \mathsf{c}\ \{Q\} \qquad P \Rightarrow P'}{\{P\}\ \mathsf{c}\ \{Q\}}\ \text{[R-STRENGTHEN-PRE]}$$

$$\frac{\dfrac{}{\{true\}\ \mathsf{y := x}\ \{\mathsf{y = x}\}} \qquad \mathsf{y} = 10 \Rightarrow true}{\{\mathsf{y} = 10\}\ \mathsf{y := x}\ \{\mathsf{y = x}\}}$$

POST-CONDITION WEAKENING

$$\frac{\{P\}\ \mathsf{c}\ \{Q'\} \qquad Q' \Rightarrow Q}{\{P\}\ \mathsf{c}\ \{Q\}}\ \text{[R-WEAKEN-POST]}$$

POST-CONDITION WEAKENING

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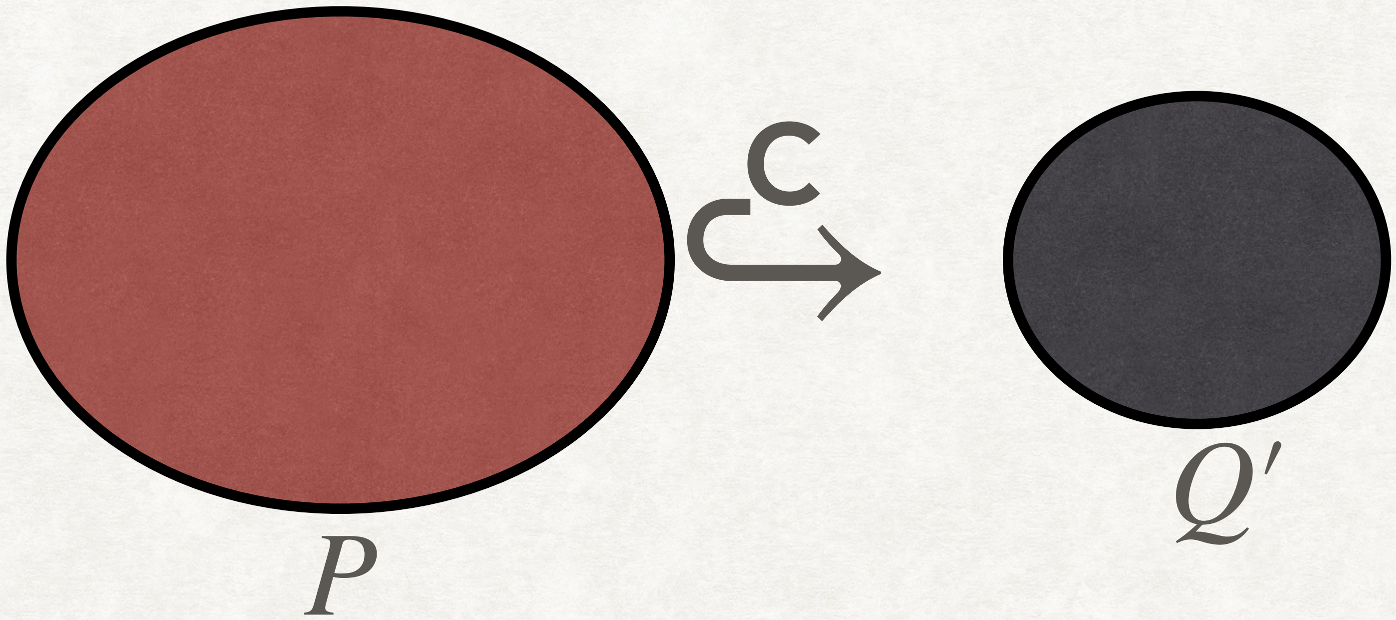
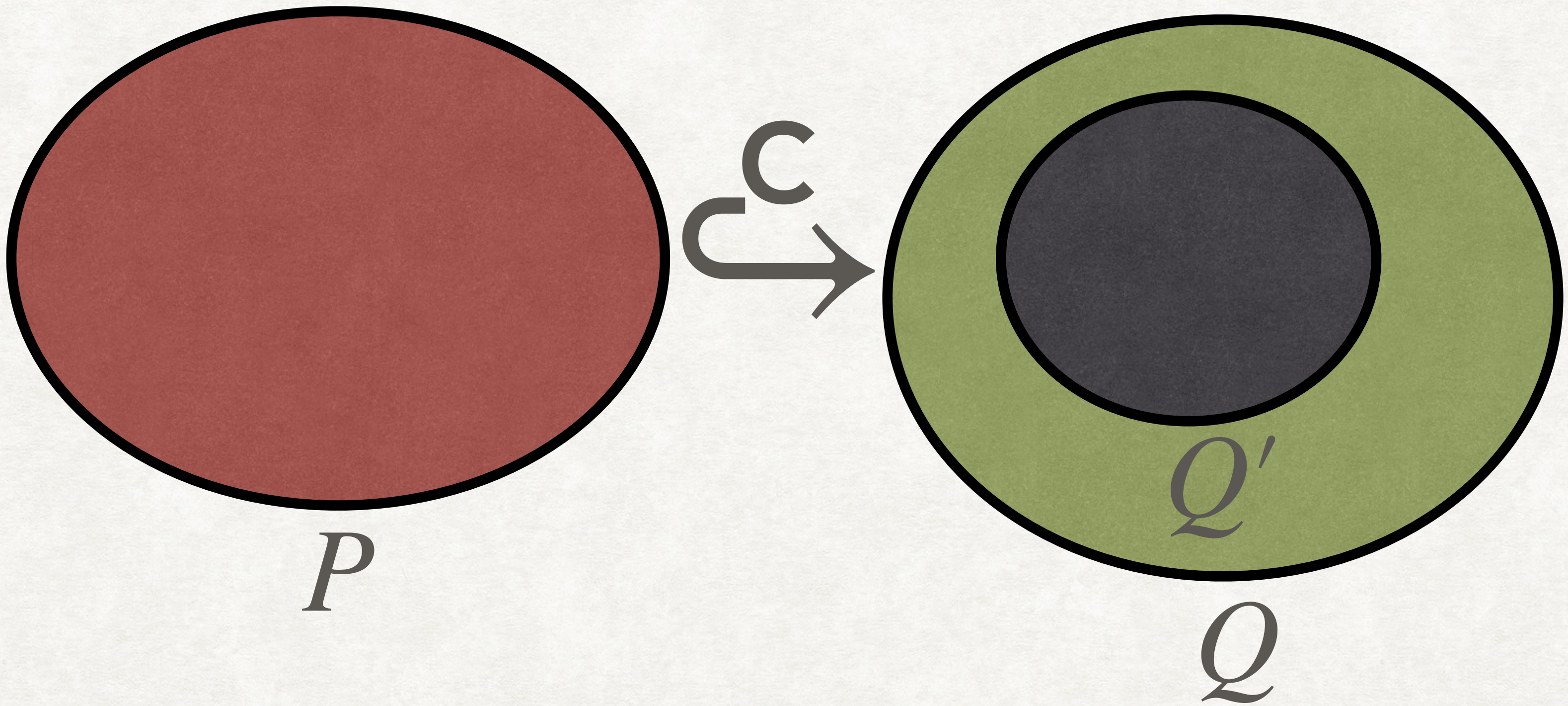


POST-CONDITION WEAKENING

$$\frac{\{P\}\ \mathsf{c}\ \{Q'\} \qquad Q' \Rightarrow Q}{\{P\}\ \mathsf{c}\ \{Q\}}\ \text{[R-WEAKEN-POST]}$$



INFERENCE RULES

COMPOUND STATEMENTS

$$\frac{\{P\}\ \mathsf{c}_1\ \{R\} \qquad \{R\}\ \mathsf{c}_2\ \{Q\}}{\{P\}\ \mathsf{c}_1; \mathsf{c}_2\ \{Q\}} \quad \text{[R-SEQ]}$$

INFERENCE RULES

COMPOUND STATEMENTS

$$\frac{\{P\}\ \mathsf{c}_1\ \{R\} \qquad \{R\}\ \mathsf{c}_2\ \{Q\}}{\{P\}\ \mathsf{c}_1; \mathsf{c}_2\ \{Q\}}$$

[R-SEQ]

$$\frac{\{P \wedge F\}\ \mathsf{c}_1\ \{Q\} \qquad \{P \wedge \neg F\}\ \mathsf{c}_2\ \{Q\}}{\{P\}\ \text{if}\ (F)\ \text{then}\ \mathsf{c}_1\ \text{else}\ \mathsf{c}_2\ \{Q\}}$$

[R-IF-THEN-ELSE]

INFERENCE RULES

COMPOUND STATEMENTS

$$\frac{\{P\}\ \mathsf{c}_1\ \{R\} \qquad \{R\}\ \mathsf{c}_2\ \{Q\}}{\{P\}\ \mathsf{c}_1; \mathsf{c}_2\ \{Q\}}$$

[R-SEQ]

$$\frac{\{P \wedge F\}\ \mathsf{c}_1\ \{Q\} \qquad \{P \wedge \neg F\}\ \mathsf{c}_2\ \{Q\}}{\{P\}\ \text{if } (F) \text{ then } \mathsf{c}_1 \text{ else } \mathsf{c}_2\ \{Q\}}$$

[R-IF-THEN-ELSE]

Prove This!

SEQUENCING

EXAMPLE

$$\frac{\{P\}\ \mathsf{c}_1\ \{R\} \qquad \{R\}\ \mathsf{c}_2\ \{Q\}}{\{P\}\ \mathsf{c}_1; \mathsf{c}_2\ \{Q\}}\ \text{[R-SEQ]}$$

$$\frac{}{\{true\}\ \mathsf{x} := 2;\ \mathsf{y} := \mathsf{x}\ \{\mathsf{y} = 2 \land \mathsf{x} = 2\}}$$

SEQUENCING

EXAMPLE

$$\frac{\{P\}\ \mathsf{c}_1\ \{R\} \qquad \{R\}\ \mathsf{c}_2\ \{Q\}}{\{P\}\ \mathsf{c}_1;\mathsf{c}_2\ \{Q\}}\ \text{[R-SEQ]}$$

$$\frac{\dfrac{}{\{true\}\ \mathsf{x} := 2\ \{\mathsf{x} = 2\}} \qquad \dfrac{}{\{\mathsf{x} = 2\}\ \mathsf{y} := \mathsf{x}\ \{\mathsf{y} = 2 \wedge \mathsf{x} = 2\}}}{\{true\}\ \mathsf{x} := 2;\ \mathsf{y} := \mathsf{x}\ \{\mathsf{y} = 2 \wedge \mathsf{x} = 2\}}$$

IF-THEN-ELSE

EXAMPLE

$$\frac{\{P \wedge F\}\ \mathsf{c}_1\ \{Q\} \qquad \{P \wedge \neg F\}\ \mathsf{c}_2\ \{Q\}}{\{P\}\ \text{if } (F) \text{ then } \mathsf{c}_1 \text{ else } \mathsf{c}_2\ \{Q\}}\ \text{[R-IF-THEN-ELSE]}$$

$$\frac{}{\{true\}\ \text{if (x > 0) then y := x else y := -x}\{\text{y} \geq 0\}}$$

IF-THEN-ELSE

EXAMPLE

$$\frac{\{P \wedge F\}\ c_1\ \{Q\} \qquad \{P \wedge \neg F\}\ c_2\ \{Q\}}{\{P\}\ \text{if } (F) \text{ then } c_1 \text{ else } c_2\ \{Q\}}\ \text{[R-IF-THEN-ELSE]}$$

$$\frac{\dfrac{\dfrac{}{\{x \geq 0\}\ y := x\ \{y \geq 0\}} \qquad x > 0 \Rightarrow x \geq 0}{\{x > 0\}\ y := x\ \{y \geq 0\}} \qquad \dfrac{}{\{x \leq 0\}\ y := -x\ \{y \geq 0\}}}{\{true\}\ \text{if } (x > 0) \text{ then } y := x \text{ else } y := -x \{y \geq 0\}}$$

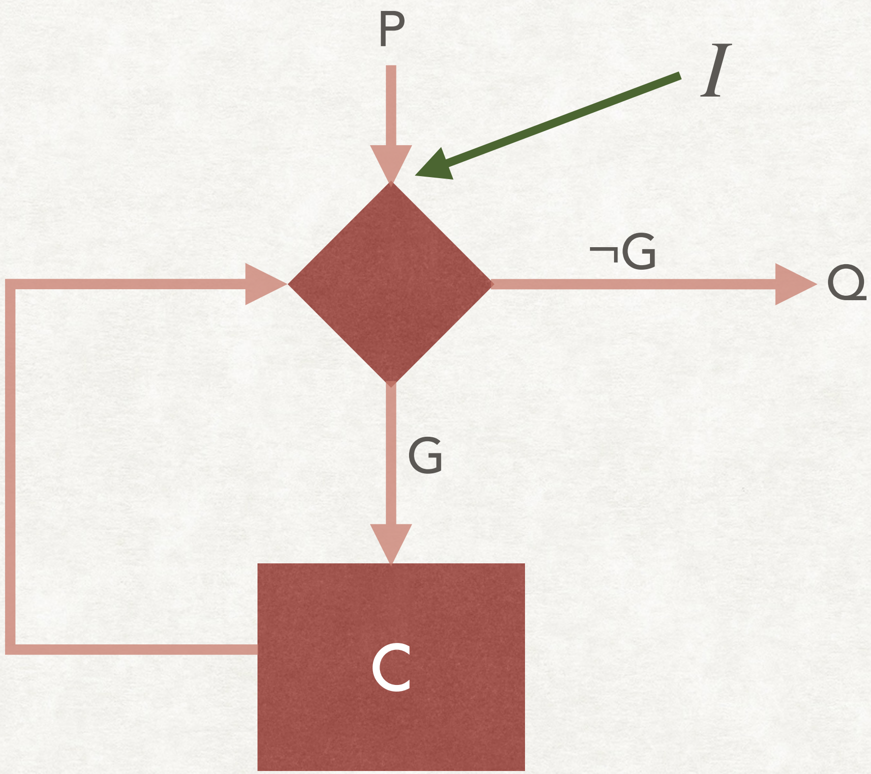
WHILE LOOPS

LOOP INVARIANTS

- Our goal is to prove the validity of $\{P\}$ while(G) do c $\{Q\}$
  - Both sp and wp lead to non-terminating procedures.
- We will instead assume a loop invariant I which is an over-approximation of the states possible during execution of the loop, but is sufficient enough to prove the Hoare Triple.
  - I is assumed to be provided by the programmer.
- Hoare Logic provides inference rules to prove that I is indeed a loop invariant.

WHILE LOOPS

LOOP INVARIANTS



- I needs to satisfy three properties:
  - I must hold initially at the start of the loop.
  - I must hold at the end of every iteration of the loop.
  - After exiting from the loop, I must imply the post-condition.

WHILE LOOPS

LOOP INVARIANTS

- Consider the code
    - `i:=0; j:=0; while(i<n) do i:=i+1; j:=i+j;`
- Which of the following are loop invariants?
    - $i \leq n$
    - $i < n$
    - $i \geq 0$

WHILE LOOP

INFERENCE RULE

$$\frac{\{I \land F\}\ \text{c}\ \{I\}}{\{I\}\ \text{while(F) do c;}\ \{I \land \neg F\}}\ \text{[R-WHILE-1]}$$

WHILE LOOP

INFERENCE RULE

$$\frac{\{I \wedge F\}\ \mathsf{c}\ \{I\}}{\{I\}\ \text{while(F) do c;}\ \{I \wedge \neg F\}}\ \text{[R-WHILE-1]}$$

$$\frac{P \Rightarrow I \qquad \{I \wedge F\}\ \mathsf{c}\ \{I\} \qquad I \wedge \neg F \Rightarrow Q}{\{P\}\ \text{while(F) do c;}\ \{Q\}}\ \text{[R-WHILE-2]}$$

WHILE LOOP

INFERENCE RULE - EXAMPLE

Prove $\{i = 0 \wedge n > 0\}$ while(i < n) do i := i+1; $\{i = n\}$

Loop Invariants: $i \geq 0$, $i \leq n$, $n > 0\ldots$

Which loop invariant is useful for proving the Hoare Triple?

$I \triangleq i \leq n$

WHILE LOOP

INFERENCE RULE - EXAMPLE

Prove $\{i = 0 \wedge n > 0\}$ while(i < n) do i := i+1; $\{i = n\}$

Loop Invariants: $i \geq 0$, $i \leq n$, $n > 0 \ldots$

Which loop invariant is useful for proving the Hoare Triple?

$$I \triangleq i \leq n$$

$\{i = 0 \wedge n > 0\}$ while(i < n) do i := i+1; $\{i = n\}$

WHILE LOOP

INFERENCE RULE - EXAMPLE

Prove $\{i = 0 \land n > 0\}$ while(i < n) do i := i+1; $\{i = n\}$

Loop Invariants: $i \geq 0$, $i \leq n$, $n > 0 \ldots$

Which loop invariant is useful for proving the Hoare Triple?

$$I \triangleq i \leq n$$

$$\dfrac{\{i = 0 \land n > 0\} \Rightarrow i \leq n \qquad \dfrac{\dfrac{}{\{i \leq n \land i < n\}\text{i:=i+1;}\{i \leq n\}}}{\{i \leq n \land i < n\}\text{i:=i+1;}\{i \leq n\}} \qquad i \leq n \land i \geq n \Rightarrow i = n}{\{i = 0 \land n > 0\}\ \text{while(i < n) do i := i+1;}\ \{i = n\}}$$

LOOP INVARIANT VS INDUCTIVE LOOP INVARIANT

- Consider again the code
  - `i:=0; j:=0; while(i<n) do i:=i+1; j:=i+j;`
- Is j ≥ 0 a loop invariant?
  - Yes, it does hold at the beginning and at the end of every iteration.
- Does $\{j \geq 0 \land i < n\}$i:=i+1; j:=i+j;$\{j \geq 0\}$ hold?
  - NO! j ≥ 0 is not an inductive loop invariant.
  - The inference rule admits only inductive loop invariants.
- How to strengthen the invariant to make it inductive?
  - $j \geq 0 \land i \geq 0$ is an inductive loop invariant.

COMPLETE EXAMPLE

$\{n > 0\}$

```
i := 0;
j := 0;
while(i < n) do
    i := i + 1;
    j := i + j;
```

$\{2j = n(n+1)\}$

COMPLETE EXAMPLE

$\{n > 0\}$

$\{P_1\}$

```
i := 0;
```

$\{P_2\}$

```
j := 0;
```

$\{P_3\}$

```
while(i < n) do
```

$\{P_4\}$

```
    i := i + 1;
```

$\{P_5\}$

```
    j := i + j;
```

$\{P_6\}$

$\{P_7\}$

$\{2j = n(n+1)\}$

Loop Invariant:
???

COMPLETE EXAMPLE

$\{n > 0\}$

$\{P_1\}$

```
i := 0;
```

$\{P_2\}$

```
j := 0;
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$\{P_3\}$

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while(i < n) do
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    i := i + 1;
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$\{P_5\}$

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    j := i + j;
```

$\{P_6\}$

$\{P_7\}$

$\{2j = n(n+1)\}$

Loop Invariant:

$2j = i(i+1) \wedge i \leq n$

COMPLETE EXAMPLE

$\{n > 0\}$

$\{P_1\}$

```
i := 0;
```

$\{P_2\}$

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j := 0;
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$\{2j = i(i+1) \wedge i \leq n\}$

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while(i < n) do
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$\{2j = i(i+1) \wedge i \leq n \wedge i < n\}$

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    i := i + 1;
```

$\{P_5\}$

```
    j := i + j;
```

$\{2j = i(i+1) \wedge i \leq n\}$

$\{2j = i(i+1) \wedge i \leq n \wedge \neg(i < n)\}$

$\{2j = n(n+1)\}$

$$\frac{\{I \wedge F\}\ \mathsf{c}\ \{I\}}{\{I\}\ \mathsf{while(F)\ do\ c;}\ \{I \wedge \neg F\}}$$

COMPLETE EXAMPLE

$\{n > 0\}$

$\{P_1\}$

```
i := 0;
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$\{P_2\}$

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$\{2j = i(i+1) \land i \leq n\}$

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while(i < n) do
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$\{2j = i(i+1) \land i \leq n \land i < n\}$

```
    i := i + 1;
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$\{P_5\}$

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    j := i + j;
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$\{2j = i(i+1) \land i \leq n\}$

$\{2j = i(i+1) \land i \leq n \land \neg(i < n)\}$

$\{2j = n(n+1)\}$

$$\frac{\{P\}\ c\ \{Q'\} \qquad Q' \Rightarrow Q}{\{P\}\ c\ \{Q\}}$$

[R-WEAKEN-POST]

$2j = i(i+1) \land i \leq n \land \neg(i < n)$

$\Rightarrow 2j = n(n+1)$

COMPLETE EXAMPLE

$\{n > 0\}$

$\{P_1\}$

```
i := 0;
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$\{P_2\}$

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j := 0;
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$\{2j = i(i+1) \wedge i \leq n\}$

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while(i < n) do
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    i := i + 1;
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$\{P_5\}$

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    j := i + j;
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$\{2j = i(i+1) \wedge i \leq n\}$

$\{2j = n(n+1)\}$

$$\frac{\{P\}\ \mathsf{c}\ \{Q'\} \qquad Q' \Rightarrow Q}{\{P\}\ \mathsf{c}\ \{Q\}}$$

[R-WEAKEN-POST]

$2j = i(i+1) \wedge i \leq n \wedge \neg(i < n)$

$\Rightarrow 2j = n(n+1)$

COMPLETE EXAMPLE

$\{n > 0\}$

$\{P_1\}$

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i := 0;
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$\{P_2\}$

```
j := 0;
```

$\{2j = i(i+1) \wedge i \leq n\}$

```
while(i < n) do
```

$\{2j = i(i+1) \wedge i \leq n \wedge i < n\}$

```
    i := i + 1;
```

$\{2i + 2j = i(i+1) \wedge i \leq n\}$

```
    j := i + j;
```

$\{2j = i(i+1) \wedge i \leq n\}$

$\{2j = n(n+1)\}$

$$\{P[e/x]\}\ x := e\ \{P\}$$

[R-ASSIGN]

$(2j = i(i+1) \wedge i \leq n)[i + j/j]$

$\equiv 2(i + j) = i(i+1) \wedge i \leq n$

$\equiv 2i + 2j = i(i+1) \wedge i \leq n$

COMPLETE EXAMPLE

$\{n > 0\}$

$\{P_1\}$

```
i := 0;
```

$\{P_2\}$

```
j := 0;
```

$\{2j = i(i+1) \wedge i \leq n\}$

```
while(i < n) do
```

$\{2j = i(i+1) \wedge i \leq n \wedge i < n\}$

$\{2j = i(i+1) \wedge i + 1 \leq n\}$

```
    i := i + 1;
```

$\{2i + 2j = i(i+1) \wedge i \leq n\}$

```
    j := i + j;
```

$\{2j = i(i+1) \wedge i \leq n\}$

$\{2j = n(n+1)\}$

$$\frac{}{\{P[e/x]\}\ x := e\ \{P\}}\ \text{[R-ASSIGN]}$$

$(2i + 2j = i(i+1) \wedge i \leq n)[(i+1)/i]$

$\equiv 2(i+1) + 2j = (i+1)(i+2) \wedge i + 1 \leq n$

$\equiv 2j = i(i+1) \wedge i + 1 \leq n$

COMPLETE EXAMPLE

$\{n > 0\}$

$\{P_1\}$

```
i := 0;
```

$\{P_2\}$

```
j := 0;
```

$\{2j = i(i+1) \wedge i \leq n\}$

```
while(i < n) do
```

$\{2j = i(i+1) \wedge i \leq n \wedge i < n\}$

$\{2j = i(i+1) \wedge i + 1 \leq n\}$

```
    i := i + 1;
```

$\{2i + 2j = i(i+1) \wedge i \leq n\}$

```
    j := i + j;
```

$\{2j = i(i+1) \wedge i \leq n\}$

$\{2j = n(n+1)\}$

$$2j = i(i+1) \wedge i \leq n \wedge i < n$$
$$\Rightarrow 2j = i(i+1) \wedge i + 1 \leq n$$

COMPLETE EXAMPLE

$\{n > 0\}$

$\{P_1\}$

```
i := 0;
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    j := i + j;
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$\{2j = i(i+1) \wedge i \leq n\}$

$\{2j = n(n+1)\}$

$$2j = i(i+1) \wedge i \leq n \wedge i < n$$
$$\Rightarrow 2j = i(i+1) \wedge i + 1 \leq n$$

COMPLETE EXAMPLE

$\{n > 0\}$

$\{P_1\}$

```
i := 0;
```

$\{i(i+1) = 0 \wedge i \leq n\}$

```
j := 0;
```

$\{2j = i(i+1) \wedge i \leq n\}$

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while(i < n) do
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$\{2j = i(i+1) \wedge i \leq n \wedge i < n\}$

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    i := i + 1;
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```
    j := i + j;
```

$\{2j = i(i+1) \wedge i \leq n\}$

$\{2j = n(n+1)\}$

$$\frac{}{\{P[e/x]\}\ x := e\ \{P\}}$$

[R-ASSIGN]

COMPLETE EXAMPLE

$\{n > 0\}$

$\{n \geq 0\}$

```
i := 0;
```

$\{i(i+1) = 0 \wedge i \leq n\}$

```
j := 0;
```

$\{2j = i(i+1) \wedge i \leq n\}$

```
while(i < n) do
```

$\{2j = i(i+1) \wedge i \leq n \wedge i < n\}$

```
    i := i + 1;
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$\{2i + 2j = i(i+1) \wedge i \leq n\}$

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    j := i + j;
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$\{2j = i(i+1) \wedge i \leq n\}$

$\{2j = n(n+1)\}$

$$\frac{}{\{P[e/x]\}\ x := e\ \{P\}}$$

[R-ASSIGN]

COMPLETE EXAMPLE

$\{n > 0\}$

```
i := 0;
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$\{i(i+1) = 0 \wedge i \leq n\}$

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j := 0;
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$\{2j = i(i+1) \wedge i \leq n\}$

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while(i < n) do
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$\{2j = i(i+1) \wedge i \leq n\}$

$\{2j = n(n+1)\}$

$$\frac{\{P'\}\ \mathtt{c}\ \{Q\} \qquad P \Rightarrow P'}{\{P\}\ \mathtt{c}\ \{Q\}}$$

[R-STRENGTHEN-PRE]

SOUNDNESS AND COMPLETENESS

- All the inference rules together provide a procedure for establishing a Hoare triple $\{P\}c\{Q\}$.
- Soundness: If we can establish $\{P\}c\{Q\}$ using the inference rules, then is $\{P\}c\{Q\}$ a valid Hoare Triple?
  - Yes.
- Completeness: If $\{P\}c\{Q\}$ is a valid Hoare Triple, then can we always use the inference rules to establish it?
  - Relatively Complete.
  - If the underlying FOL theory is complete, then Hoare Logic is complete.

HOARE LOGIC

VERIFICATION CONDITION GENERATION

- We have already seen that the weakest pre-condition operator can be used to prove Hoare Triples:
  - $\{P\}c\{Q\}$ iff $P \Rightarrow wp(Q, c)$
- Finding exact wp for loops is hard. We will instead use the loop invariant as an approximate wp.
  - $awp(Q,$ while(F)@I do c) = I
  - Does this always hold?
- Also need to show that following side-conditions hold:
  - $\{\mathrm{I} \wedge \mathrm{F}\}\mathrm{c}\{\mathrm{I}\}$
  - $\mathrm{I} \wedge \neg\mathrm{F} \Rightarrow Q$

RELATION BETWEEN AWP AND WP

- Let us formally define *awp*:
  - $\forall \sigma \in awp(Q, c) \, . \, \forall \sigma' \, . \, (\sigma, c) \hookrightarrow^* (\sigma', skip) \rightarrow \sigma' \in Q$
  - Homework: Prove that this holds for $awp(Q,$ while(F)@I do c) = I, when the side-conditions hold.
- We defined $wp(Q, c) \triangleq \{\sigma \mid \forall \sigma' . \, (\sigma, c) \hookrightarrow^* (\sigma', \text{skip}) \rightarrow \sigma' \in Q\}$
  - $awp(Q, c) \subseteq wp(Q, c)$
- We can then use *awp* for verifying the validity of Hoare Triples:
  - If $P \Rightarrow awp(Q, c)$ then $\{P\}c\{Q\}$.

RELATION BETWEEN AWP AND WP

EXAMPLES

- $awp(i \geq 0$, while(i < n)@(i >= 0) do i := i+1;) = ???

RELATION BETWEEN AWP AND WP

EXAMPLES

- $awp(i \geq 0,$ while(i < n)@(i >= 0) do i := i+1;) = $i \geq 0$

RELATION BETWEEN AWP AND WP

EXAMPLES

- $awp(i \geq 0$, while(i < n)@(i >= 0) do i := i+1;) = $i \geq 0$
  - $wp(i \geq 0$, while(i < n)@(i >= 0) do i := i+1;) = ???

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- $awp(i \geq 0$, while(i < n)@(i >= 0) do i := i+1;) = $i \geq 0$
  - $wp(i \geq 0$, while(i < n)@(i >= 0) do i := i+1;) =$n \geq 0 \vee i \geq 0$

VC GENERATION - I

- We define $VC(Q, c)$ to collect the side-conditions needed for verifying that Q holds after execution of c.
- For while(F)@I do c, there are two side-conditions:
  - $\{\mathsf{I} \wedge \mathsf{F}\}\mathsf{c}\{\mathsf{I}\}$
  - $\mathsf{I} \wedge \neg\mathsf{F} \Rightarrow Q$
- $\{\mathsf{I} \wedge \mathsf{F}\}\mathsf{c}\{\mathsf{I}\}$ is valid if $\mathsf{I} \wedge \mathsf{F} \Rightarrow awp(\mathsf{I}, \mathsf{c})$.
  - c may contain loops, so we also need to consider $VC(\mathsf{I}, \mathsf{c})$.
- Hence,
  $VC(\mathsf{Q}, \text{while(F)@I do c}) \triangleq (\mathsf{I} \wedge \neg\mathsf{F} \Rightarrow \mathsf{Q}) \wedge (\mathsf{I} \wedge \mathsf{F} \Rightarrow awp(\mathsf{I}, \mathsf{c})) \wedge VC(\mathsf{I}, \mathsf{c})$

VC GENERATION - II

- $VC(Q, \text{x:=e}) \triangleq \top$
  - Also defined as $\top$ for all primitive program commands (assert, assume, havoc).
- $VC(Q, c_1; c_2) \triangleq ???$

VC GENERATION - II

- $VC(Q, \text{x:=e}) \triangleq true$
  - Also defined as *true* for all simple program commands (assert, assume, havoc).
- $VC(Q, c_1; c_2) \triangleq VC(Q, c_2) \wedge VC(awp(Q, c_2), c_1)$

VC GENERATION - II

- $VC(Q, \text{x:=e}) \triangleq true$
  - Also defined as *true* for all simple program commands (assert, assume, havoc).
- $VC(Q, c_1; c_2) \triangleq VC(Q, c_2) \land VC(awp(Q, c_2), c_1)$
- $VC(Q, \text{if(F) then } \mathrm{c}_1 \text{ else } \mathrm{c}_2) \triangleq ???$

VC GENERATION - II

- $VC(Q, \text{x:=e}) \triangleq true$
    - Also defined as *true* for all simple program commands (assert, assume, havoc).
- $VC(Q, c_1; c_2) \triangleq VC(Q, c_2) \wedge VC(awp(Q, c_2), c_1)$
- $VC(Q, \text{if(F) then c}_1 \text{ else c}_2) \triangleq VC(Q, \text{c}_1) \wedge VC(Q, \text{c}_2)$

VC GENERATION - III

- $awp(Q, c) \triangleq wp(Q, c)$ except for while loops, for which $awp(Q,$ while(F)@I do c) = I.
- Putting it all together, $\{P\}c\{Q\}$ is valid if the following FOL formula is valid:
  - $(P \rightarrow awp(Q, c)) \wedge VC(Q, c)$

RELATION BETWEEN AWP AND HOARE TRIPLES

- What is the relation between $awp(Q, c)$ and validity of the Hoare Triple $\{P\}c\{Q\}$?
  - Is it possible that $P \rightarrow awp(Q, c)$ is valid and $\{P\}c\{Q\}$ is not valid?
  - Is it possible that $\{P\}c\{Q\}$ is valid and $\neg(P \rightarrow awp(Q, c))$ is satisfiable?
  - How about $\neg(P \rightarrow wp(Q, c))$?

RELATION BETWEEN AWP AND WP

EXAMPLES

- $awp(i \geq 0$, while(i < n)@(i >= 0) do i := i+1;) = $i \geq 0$
  - $wp(i \geq 0$, while(i < n)@(i >= 0) do i := i+1;) =$n \geq 0 \vee i \geq 0$

VC GENERATION

SOUNDNESS AND COMPLETENESS

- Is the VC generation procedure sound?
  - Yes. Prove this!
- Is the VC generation procedure complete?
  - No. It is not even relatively complete.
  - The annotated loop invariant may not be strong enough.
- Can the VC generation procedure be fully automated?
  - Yes. Whole point of the exercise!

EXAMPLE

$\{ \top \}$

```
i := 1;
sum := 0;
while(i <= n) do
    j := 1;
    while(j <= i) do
        sum := sum + j; j := j + 1;
    i := i + 1;
```

$\{sum \geq 0\}$

EXAMPLE

```
{ ⊤ }
i := 1;
sum := 0;
while(i <= n)@(sum ≥ 0) do
    j := 1;
    while(j <= i)@(sum ≥ 0 ∧ j ≥ 0) do
        sum := sum + j; j := j + 1;
    i := i + 1;
{sum ≥ 0}
```

- $VC(sum \geq 0,$outer loop$)$:
  - $sum \geq 0 \wedge i > n \rightarrow sum \geq 0$
  - $sum \geq 0 \wedge i \leq n \rightarrow sum \geq 0 \wedge 1 \geq 0$
  - $VC(sum \geq 0,$inner loop$)$

EXAMPLE

```
{ ⊤ }
i := 1;
sum := 0;
while(i <= n)@(sum ≥ 0) do
    j := 1;
    while(j <= i)@(sum ≥ 0 ∧ j ≥ 0) do
        sum := sum + j; j := j + 1;
    i := i + 1;
{sum ≥ 0}
```

- $VC(sum \geq 0$,inner loop):
  - $sum \geq 0 \land j \geq 0 \land j > i \rightarrow sum \geq 0$
  - $sum \geq 0 \land j \geq 0 \land j \leq i \rightarrow sum + j \geq 0 \land j + 1 \geq 0$

EXAMPLE

$\{ \top \}$

```
i := 1;
sum := 0;
while(i <= n)@(sum ≥ 0) do
    j := 1;
    while(j <= i)@(sum ≥ 0 ∧ j ≥ 0) do
        sum := sum + j; j := j + 1;
    i := i + 1;
```

$\{sum \geq 0\}$

- Final Formula:
    - $\top \rightarrow 0 \geq 0 \wedge VC(sum \geq 0,$outer loop$)$

ANNOUNCEMENTS

- Tutorial session on Dafny tomorrow, conducted by Sheera Shamsu.
  - Tool Assignment-2 will be based on Dafny.
- Theory Assignment-2 will be released next week.
- We will start project meetings soon.

ADDING FUNCTIONS TO IMP

$$
\begin{aligned}
&\mathtt{p} = \mathtt{l}^* \\
&\mathtt{l} = \mathtt{function}\ f(\mathtt{x}_1, \ldots, \mathtt{x}_n)\{\mathtt{c}\} \\
&\mathtt{c} = \mathtt{x} := \mathtt{exp} \mid \mathtt{x} := \mathtt{havoc} \\
&\quad = \mid \mathtt{assume(F)} \mid \mathtt{assert(F)} \\
&\quad = \mid \mathtt{skip} \mid \mathtt{c;c} \mid \mathtt{if(F)\ then\ c\ else\ c} \mid \mathtt{while(F)\ do\ c} \\
&\quad = \mid \mathtt{x} := f(\mathtt{exp}_1, \ldots, \mathtt{exp}_n) \mid \mathtt{return\ exp}
\end{aligned}
$$

- We will assume that all variables are local, and local variables across different function instances are disjoint.
- We will also assume that formal parameters are not modified by a function.

MODULAR VERIFICATION

- Each function is annotated with a pre-condition and a post-condition.
- Pre-condition specifies what is expected of the function's arguments
  - Formula in FOL whose free variables are the formal parameters of the function.
- Post-condition describes the function's return value
  - Formula in FOL whose free variables are the formal parameters and a special variable called ret.
- Together, pre-condition and post-condition specify a *contract.*
  - If the function is called with values which obey the pre-condition, then the output of the function will obey the post-condition.

VERIFYING FUNCTION CONTRACT

```
function f(x1,…,xn)
    requires(Pre)
    ensures(Post)
    {Body;}
```

- The function contract can be verified by proving the validity of the Hoare Triple $\{Pre\}\ Body\ \{Post\}$

VERIFYING FUNCTION CALLS

- The function body may have calls to other functions (or even itself)
  - $\{P\}x := f(e_1, \ldots, e_n)\{Q\}$
- If we can guarantee that the function's pre-condition holds before the call, then we can assume that the function's post-condition will hold after the call.
- We model the function call as follows:

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- We model the function call as follows:

```
assert(Pre[e1/x1,…,en/xn]);
assume(Post[tmp/ret,e1/x1,…,en/xn]);
x := tmp;
```

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- Why do we have to use tmp?
- What is the generated VC?

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x := tmp;
```

- Why do we have to use tmp?
- What is the generated VC? $P \rightarrow (Pre \wedge (Post \rightarrow Q[tmp/y]))$

EXAMPLE

```
FindMax(a,l,u)
  requires(l >= 0 && l <= u && u < |a|)
  ensures(∀i. l<=i<=u → ret >= a[i])
  {
     if (l == u)
        return a[l];
     else
        m := FindMax(a,l+1,u);
        if (a[l] > m)
           return a[l];
        else
           return m;
  }
```

EXAMPLE

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FindMax(a,l,u)
  requires(l >= 0 && l <= u && u < |a|)
  ensures(∀i. l<=i<=u → ret >= a[i])
  {
     if (l == u)
        return a[l];
     else
        assert(Pre[l+1/l]);
        assume(Post[tmp/ret,l+1/l]);
        m := tmp;
        if (a[l] > m)
           return a[l];
        else
           return m;
  }
```

EXAMPLE

$\{l \geq 0 \wedge l \leq u \wedge u < |a|\}$

```
 if (l == u)
    ret:=a[l];
 else
    assert(Pre[l+1/l]);
    assume(Post[tmp/ret,l+1/l]);
    m := tmp;
    if (a[l] > m)
       ret:=a[l];
    else
       ret:=m;
```

$\{\forall i\,.\, l \leq i \leq u \rightarrow ret \geq a[i]\}$

$$
\begin{aligned}
Pre \rightarrow\ & (l = u \rightarrow Post[a[l]/ret]) \wedge \\
& l \neq u \rightarrow Pre[(l+1)/l] \\
& \wedge Post[tmp/ret, (l+1)/l] \rightarrow \\
& (a[l] > tmp \rightarrow Post[a[l]/ret]) \wedge (a[l] \leq tmp \rightarrow Post[tmp/ret])
\end{aligned}
$$

EXAMPLE - BINARY SEARCH

```
BinarySearch(a,l,u,e)
  requires(l >= 0 && u < |a|)
  ensures(ret ↔ ∃i.l <= i <= u & a[i] == e)
  {
    if (l > u) then
      return false;
    else
    {
      m := (l+u)/2;
      if (a[m]==e) then
        return true;
      else
      {
        if (a[m] < e)
          return BinarySearch(a,m+1,u,e);
        else
          return BinarySearch(a,l,m-1,e);
      }
    }
  }
```

EXAMPLE - BINARY SEARCH

```
BinarySearch(a,l,u,e)
  requires(l >= 0 && u < |a| && sorted(a,l,u) )
  ensures(ret ↔ ∃i.l <= i <= u & a[i] == e)
  {
    if (l > u) then
      return false;
    else
    {
      m := (l+u)/2;
      if (a[m]==e) then
        return true;
      else
      {
        if (a[m] < e)
          return BinarySearch(a,m+1,u,e);
        else
          return BinarySearch(a,l,m-1,e);
      }
    }
  }
```

$$sorted(a, l, u) \Leftrightarrow \forall i, j \,.\, l \leq i \leq j \leq u \rightarrow a[i] \leq a[j]$$

EXAMPLE - BINARY SEARCH

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BinarySearch(a,l,u,e)
  requires(l >= 0 && u < |a| && sorted(a,l,u) )
  ensures(ret ↔ ∃i.l <= i <= u & a[i] == e)
  {
    if (l > u) then
      return false;
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    {
      m := (l+u)/2;
      if (a[m]==e) then
        return true;
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      {
        if (a[m] < e)
          return BinarySearch(a,m+1,u,e);
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          return BinarySearch(a,l,m-1,e);
      }
    }
  }
```

$$sorted(a,l,u) \Leftrightarrow \forall i,j \,.\, l \leq i \leq j \leq u \rightarrow a[i] \leq a[j]$$

BM CHAPTER 5 CONTAINS THE COMPLETE EXAMPLE

IN THE BOOK…

- More Examples (Chapters 5,6)
  - Linear Search
  - Bubble Sort
  - Quick Sort
- A slightly different VC generation procedure
- Heuristics for crafting loop invariants

HANDLING GLOBAL VARIABLES

- If there are global variables shared across functions, then executing a function can cause side effects.
  - Is the previous approach still sound?
- We will use havoc assignments to model side-effects.
- Function contracts now specify global variables which may be modified.

```
function f(x1,…,xn)
    requires(Pre)
    ensures(Post)
    modifies(v1,…,vm)
    {Body;}
```

HANDLING GLOBAL VARIABLES

- How to check correctness of the function contract?
- $y := f(e_1, \ldots, e_n)$ is replaced by

```
assert(Pre[e1/x1,…,en/xn]);
v1:=havoc;… vm:=havoc;
assume(Post[tmp/ret,e1/x1,…,en/xn]);
y := tmp;
```

ADDING POINTERS TO IMP

- We add two more program statements:
  - x := *y
  - *x := e
- Consider the following code:
  - $\{\ \top\ \}$x := y; *y := 3; *x := 2; z := *y;$\{z = 3\}$
  - Does it satisfy the specification? What is $wp(z = 3, c)$?
- We need new rules for assignment statements involving pointers.

HANDLING POINTERS

- We treat the memory as a giant array M, with the pointer variables behaving as indices into the array.
  - x := *y becomes x := M[y]
  - *x := e becomes M := M⟨x◃e⟩
- $\{???\}x := *y\{Q\}$

HANDLING POINTERS

- We treat the memory as a giant array M, with the pointer variables behaving as indices into the array.
  - x := *y becomes x := M[y]
  - *x := e becomes M := M$\langle$x$\triangleleft$e$\rangle$
- $\{Q[M[y]/x]\}x := *\, y\{Q\}$

Adapted from Isil Dillig's Lectures

HANDLING POINTERS

- We treat the memory as a giant array M, with the pointer variables behaving as indices into the array.
  - x := *y becomes x := M[y]
  - *x := e becomes M := M⟨x◁e⟩
- $\{Q[M[y]/x]\}x := *y\{Q\}$
- $\{???\} *x := e\{Q\}$

HANDLING POINTERS

- We treat the memory as a giant array M, with the pointer variables behaving as indices into the array.
  - x := *y becomes x := M[y]
  - *x := e becomes M := M$\langle$x$\lhd$e$\rangle$
- $\{Q[M[y]/x]\}x := *\,y\{Q\}$
- $\{Q[M\langle x \lhd e\rangle/M]\}\,*\,x := e\{Q\}$

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  - *x := e becomes M := M$\langle$x$\triangleleft$e$\rangle$
- $\{Q[M[y]/x]\}x := *\,y\{Q\}$
- $\{Q[M\langle x \triangleleft e\rangle/M]\} *\,x := e\{Q\}$
- Consider the code again:
  - { ⊤ }x := y; *y := 3; *x := 2; z := *y;$\{z = 3\}$

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- We treat the memory as a giant array M, with the pointer variables behaving as indices into the array.
  - x := *y becomes x := M[y]
  - *x := e becomes M := M⟨x◁e⟩
- $\{Q[M[y]/x]\}x := *\,y\{Q\}$
- $\{Q[M\langle x \lhd e\rangle/M]\} *\,x := e\{Q\}$
- Consider the code again:
  - { ⊤ }x := y; *y := 3; *x := 2; z := *y;$\{z = 3\}$
  - VC: ⊤ → $M\langle y \lhd 3\rangle\langle y \lhd 2\rangle[y] = 3$

Adapted from Isil Dillig's Lectures