

# STRONGEST POST-CONDITION

## SYMBOLIC EXECUTION IN THE FORWARD DIRECTION

- Given a set of states  $S$  and a command  $c$ , the strongest post-condition  $sp(S, c)$  consists of all states that can be obtained after executing  $c$  on any state in  $S$ .

$$sp(S, c) \triangleq \{\sigma' \mid \exists \sigma \in S. (\sigma, c) \hookrightarrow^* (\sigma', \text{skip})\}$$

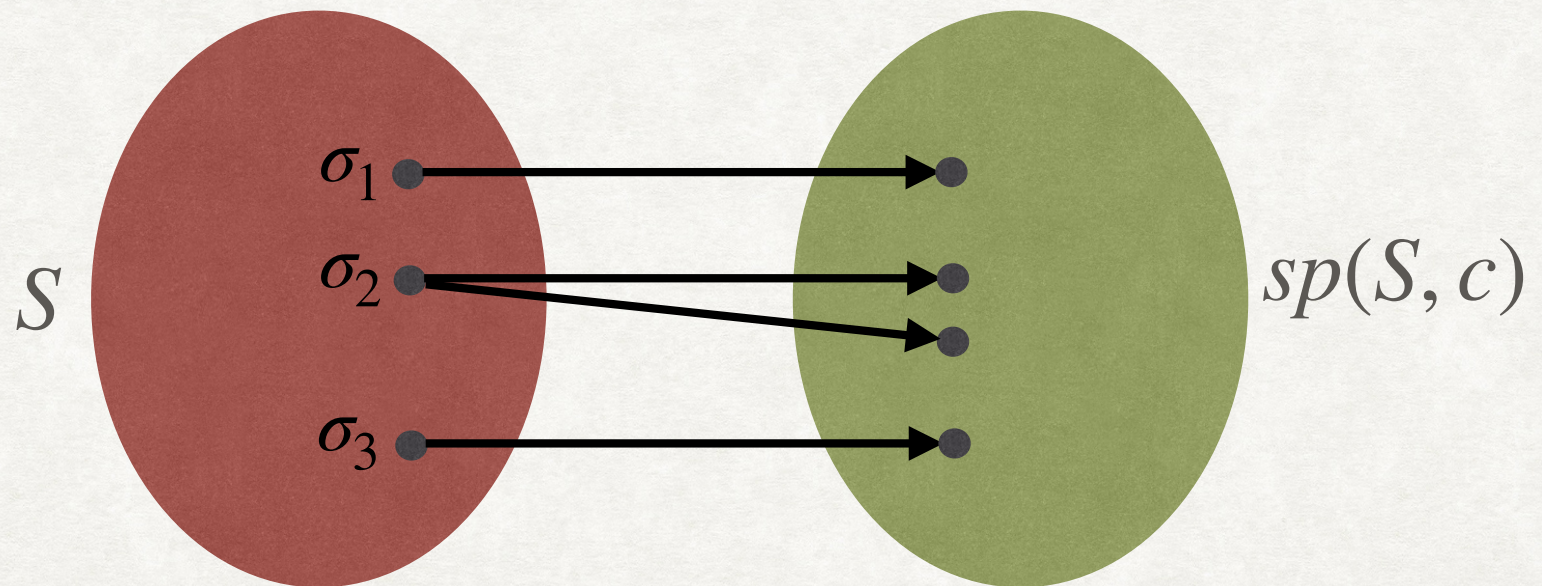
Equivalently,  $\sigma' \in sp(S, c) \leftrightarrow \exists \sigma \in S. (\sigma, c) \hookrightarrow^* (\sigma', \text{skip})$

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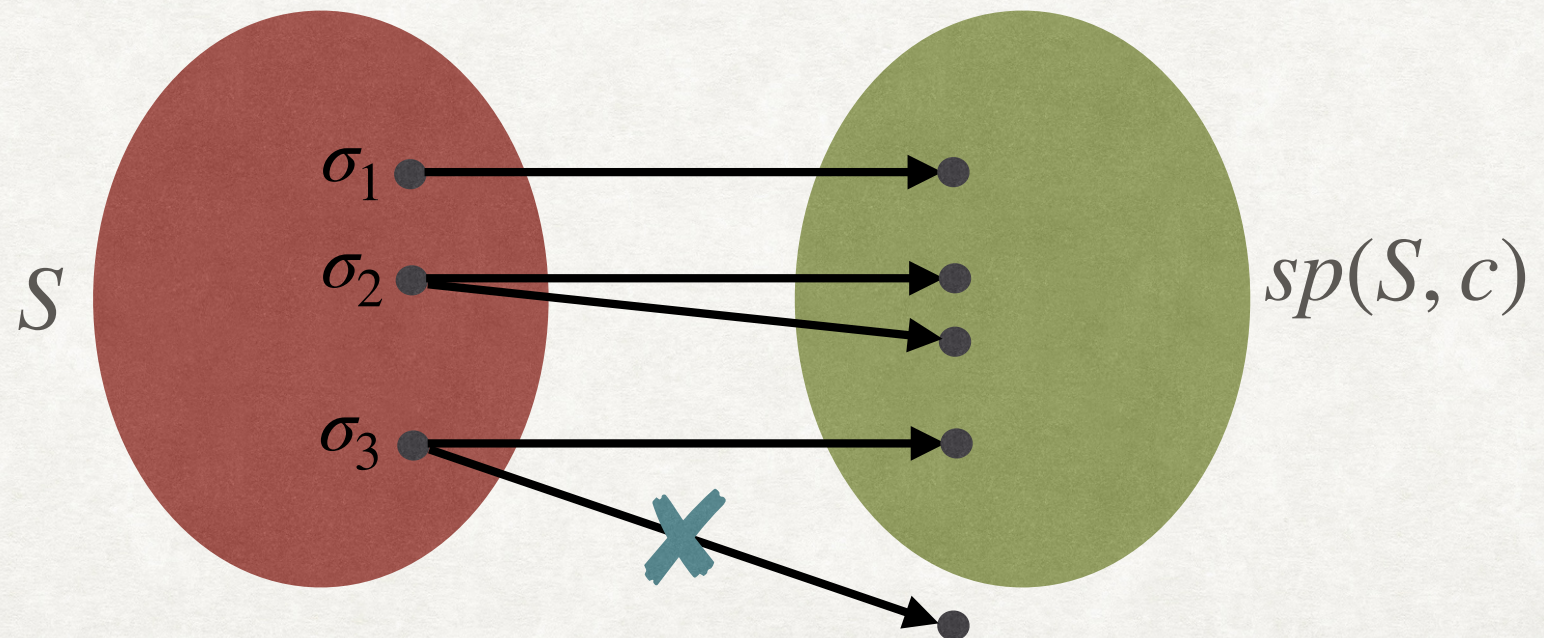


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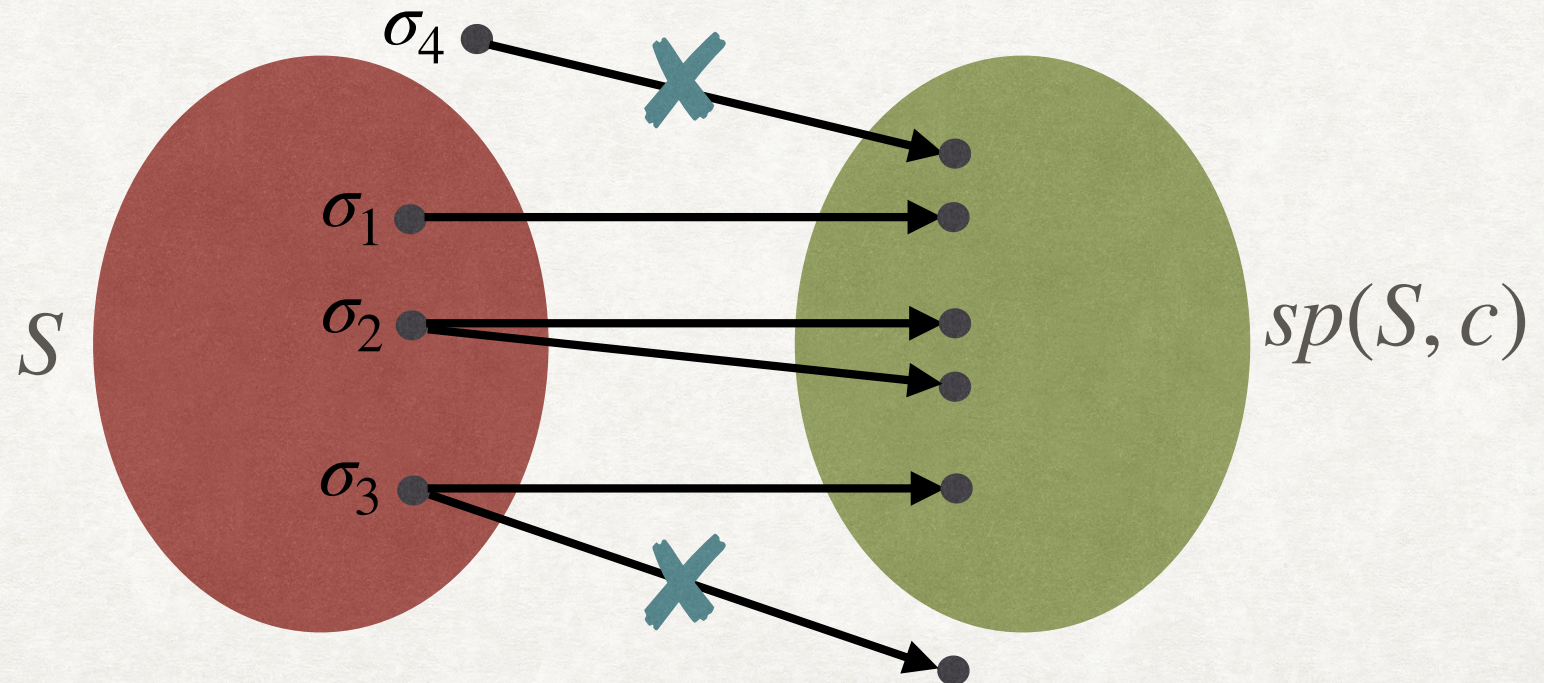


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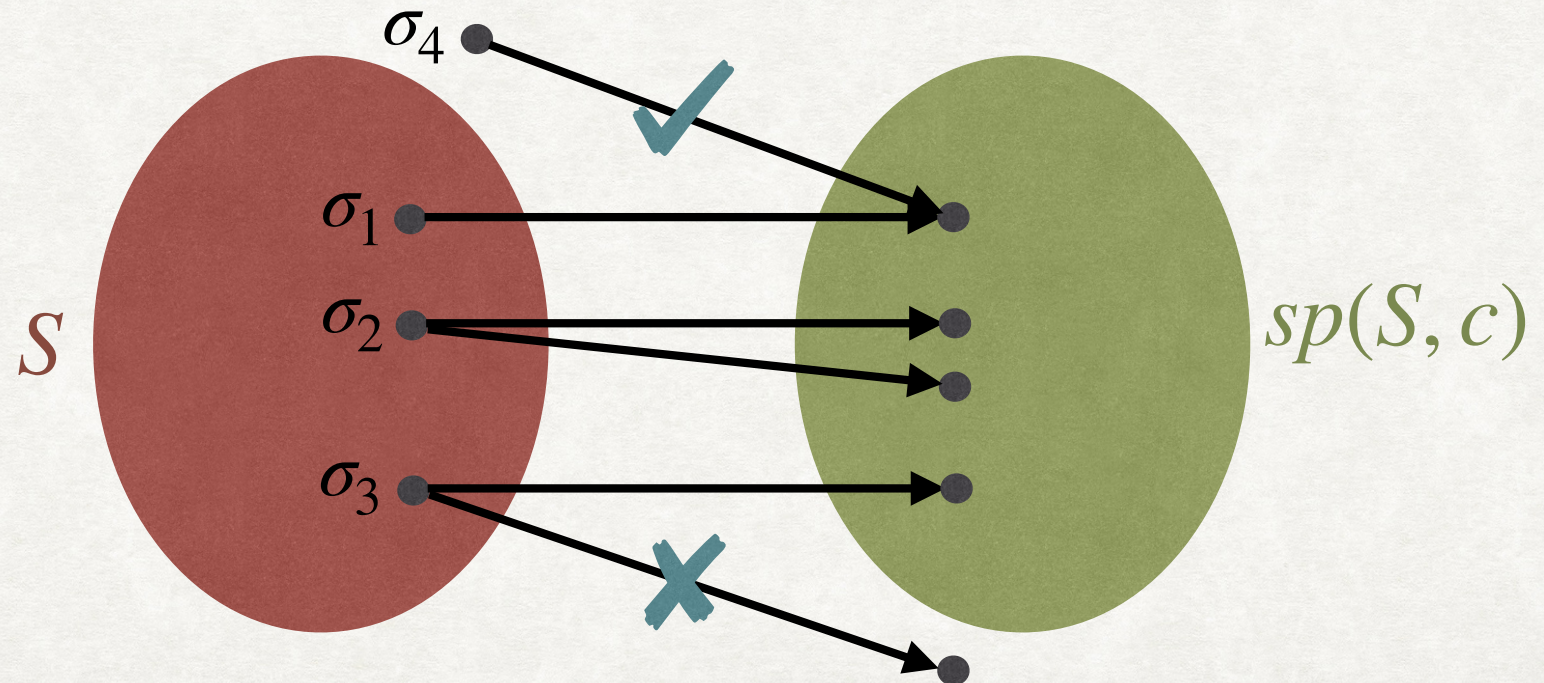


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$$sp(S, c) \triangleq \{\sigma' \mid \exists \sigma \in S. (\sigma, c) \hookrightarrow^* (\sigma', \text{skip})\}$$

- We can use a FOL formula  $F$  to represent a set of states.
- The symbolic strongest post-condition operator can be defined as:

$$\sigma' \models sp(F, c) \Leftrightarrow \exists \sigma. \sigma \models F \wedge (\sigma, c) \hookrightarrow^* (\sigma', \text{skip})$$

- We can now use the semantics in FOL ( $\rho$ ) to define symbolic  $sp$ :

$$sp(F, c) \triangleq (\exists V. F \wedge \rho(c))[V/V']$$

FIRST ELIMINATE EXISTENTIAL QUANTIFICATION  
ON  $V$ , THEN SUBSTITUTE  $V$  FOR  $V'$

# QUANTIFIER ELIMINATION

- Eliminate quantifiers in a formula  $F$  to obtain an equivalent formula  $G$  (equivalent modulo  $T_{\mathbb{Q}}$ ).
  - A decidable procedure exists for  $T_{\mathbb{Q}}$ -formulae.
  - Ferrante and Rackoff's Method (BM Chapter 7)
- Consider the formula:  $\exists y. x = y + 1$ .
  - Equivalent formula after eliminating  $y$ :  $\top$
- Consider the formula:  $\exists y. y > 1 \wedge x = 2y$ 
  - Equivalent formula after eliminating  $y$ :  $x > 2$
- What about  $\exists y. x = 2y \wedge x > y$ ?
  - Equivalent formula:  $x > 0$

# STRONGEST POST-CONDITION

## EXAMPLE

$$sp(F, c) \triangleq (\exists V. F \wedge \rho(c))[V/V']$$

Lets calculate  $sp(y > 0, x := y + 1)$



# STRONGEST POST-CONDITION

## EXAMPLE

$$sp(F, c) \triangleq (\exists V. F \wedge \rho(c))[V/V']$$

Lets calculate  $sp(y > 0, x:=y+1)$

$$\begin{aligned} sp(y > 0, x:=y+1) &\triangleq \exists x. \exists y. y > 0 \wedge \rho(x:=y+1) \\ &\equiv \exists x. \exists y. y > 0 \wedge x' = y + 1 \wedge y' = y \\ &\equiv y' > 0 \wedge x' = y' + 1 \leftarrow \\ &\equiv y > 0 \wedge x = y + 1 \leftarrow \end{aligned}$$

Eliminate  $x$  and  $y$

Substitute  $x'$  and  $y'$  with  $x$  and  $y$

# ANNOUNCEMENT

- Assignments : Late Submission Policy
  - 1 Day late : 25% Penalty
  - 2 Days late : 50% Penalty
  - Submissions after 2 days will be ignored.

# STRONGEST POST-CONDITION

## EXAMPLE

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Lets calculate  $sp(y > 0, x:=y+1)$

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Alternative Formulation for Assignment Statement:

$$sp(F, x:=e) \equiv \exists x'. F[x'/x] \wedge x = e[x'/x]$$

# STRONGEST POST-CONDITION

## MORE EXAMPLES

$sp(y > 0, x := \text{havoc}) \triangleq ???$

# STRONGEST POST-CONDITION

## MORE EXAMPLES

$$\begin{aligned} sp(y > 0, x := \text{havoc}) &\triangleq \exists x. \exists y. y > 0 \wedge y' = y \quad [\rho(x := \text{havoc}) \triangleq \text{frame}(x)] \\ &\triangleq y > 0 \end{aligned}$$

# STRONGEST POST-CONDITION

## MORE EXAMPLES

$$\begin{aligned} sp(y > 0, x := \text{havoc}) &\triangleq \exists x. \exists y. y > 0 \wedge y' = y \quad [\rho(x := \text{havoc}) \triangleq \text{frame}(x)] \\ &\triangleq y > 0 \end{aligned}$$

$$sp(F, x := \text{havoc}) \triangleq \exists x. F$$

# STRONGEST POST-CONDITION

## MORE EXAMPLES

$$\begin{aligned} sp(y > 0, x := \text{havoc}) &\triangleq \exists x. \exists y. y > 0 \wedge y' = y \\ &\triangleq y > 0 \end{aligned}$$

$$sp(F, \text{assume}(G)) \triangleq ???$$

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## MORE EXAMPLES

$$\begin{aligned} sp(y > 0, x := \text{havoc}) &\triangleq \exists x. \exists y. y > 0 \wedge y' = y \\ &\triangleq y > 0 \end{aligned}$$

$$sp(F, \text{assume}(G)) \triangleq F \wedge G$$



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$$\begin{aligned} sp(y > 0, x := \text{havoc}) &\triangleq \exists x. \exists y. y > 0 \wedge y' = y \\ &\triangleq y > 0 \end{aligned}$$

$$sp(F, \text{assume}(G)) \triangleq F \wedge G$$

$$sp(F, \text{assert}(G)) \triangleq ???$$

# STRONGEST POST-CONDITION

## MORE EXAMPLES

$$\begin{aligned} sp(y > 0, x := \text{havoc}) &\triangleq \exists x. \exists y. y > 0 \wedge y' = y \\ &\triangleq y > 0 \end{aligned}$$

$$sp(F, \text{assume}(G)) \triangleq F \wedge G$$

$$\begin{aligned} sp(F, \text{assert}(G)) &\triangleq \exists V. F \wedge (G \rightarrow \text{frame}(\emptyset)) \\ &\equiv \exists V. F \wedge (\neg G \vee \text{frame}(\emptyset)) \\ &\equiv \exists V. (F \wedge \neg G) \vee \exists V. (F \wedge \text{frame}(\emptyset)) \\ &\equiv \exists V. (F \wedge \neg G) \vee F[V'/V] \\ &\equiv (\exists V. F \wedge \neg G) \vee F \end{aligned}$$

# STRONGEST POST-CONDITION

## MORE EXAMPLES

$$\begin{aligned} sp(y > 0, x := \text{havoc}) &\triangleq \exists x. \exists y. y > 0 \wedge y' = y \\ &\triangleq y > 0 \end{aligned}$$

$$sp(F, \text{assume}(G)) \triangleq F \wedge G$$

$$sp(F, \text{assert}(G)) \triangleq (\exists V. F \wedge \neg G) \vee F$$

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$$sp(F, \text{assume}(G)) \triangleq F \wedge G$$

$$sp(F, \text{assert}(G)) \triangleq (\exists V. F \wedge \neg G) \vee F$$

$$sp(\text{false}, c) \triangleq ???$$

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## EXAMPLES

$$\begin{aligned} sp(y > 0, x := \text{havoc}) &\triangleq \exists x. \exists y. y > 0 \wedge y' = y \\ &\triangleq y > 0 \end{aligned}$$

$$sp(F, \text{assume}(G)) \triangleq F \wedge G$$

$$sp(F, \text{assert}(G)) \triangleq (\exists V. F \wedge \neg G) \vee F$$

$$sp(\text{false}, c) \triangleq \text{false}$$

# EXAMPLES

- $sp(x > 5, \text{assume}(x < 20)) \equiv ???$
- $sp(x > 5, \text{assert}(x < 0)) \equiv ???$
- $sp(x > 0, x := x + 1) \equiv ???$

# EXAMPLES

- $sp(x > 5, \text{assume}(x < 20)) \equiv x > 5 \wedge x < 20$
- $sp(x > 5, \text{assert}(x < 0)) \equiv \text{true}$
- $sp(x > 0, x := x + 1) \equiv x > 1$

# STRONGEST POST-CONDITION

## COMPOUND STATEMENTS

- $sp(F, c; c') \triangleq ???$



# STRONGEST POST-CONDITION

## COMPOUND STATEMENTS

- $sp(F, c; c') \triangleq sp(sp(F, c), c')$  (**Homework:** Prove this formally.)

# STRONGEST POST-CONDITION

## COMPOUND STATEMENTS

- $sp(F, c; c') \triangleq sp(sp(F, c), c')$
- $sp(F, \text{if}(G) \text{ then } c \text{ else } c') \triangleq ???$

# STRONGEST POST-CONDITION

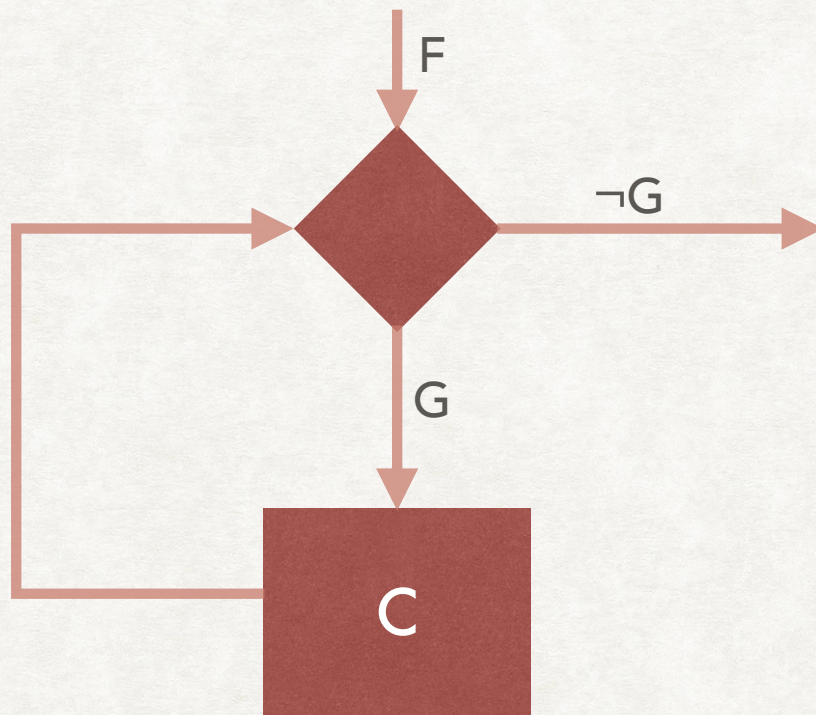
## COMPOUND STATEMENTS

- $sp(F, c; c') \triangleq sp(sp(F, c), c')$
- $sp(F, \text{if}(G) \text{ then } c \text{ else } c') \triangleq sp(F \wedge G, c) \vee sp(F \wedge \neg G, c')$  (Homework: Prove this formally.)

# STRONGEST POST-CONDITION

## WHILE LOOPS

- How to find  $sp(F, \text{while}(G) \text{ do } c)$ ?

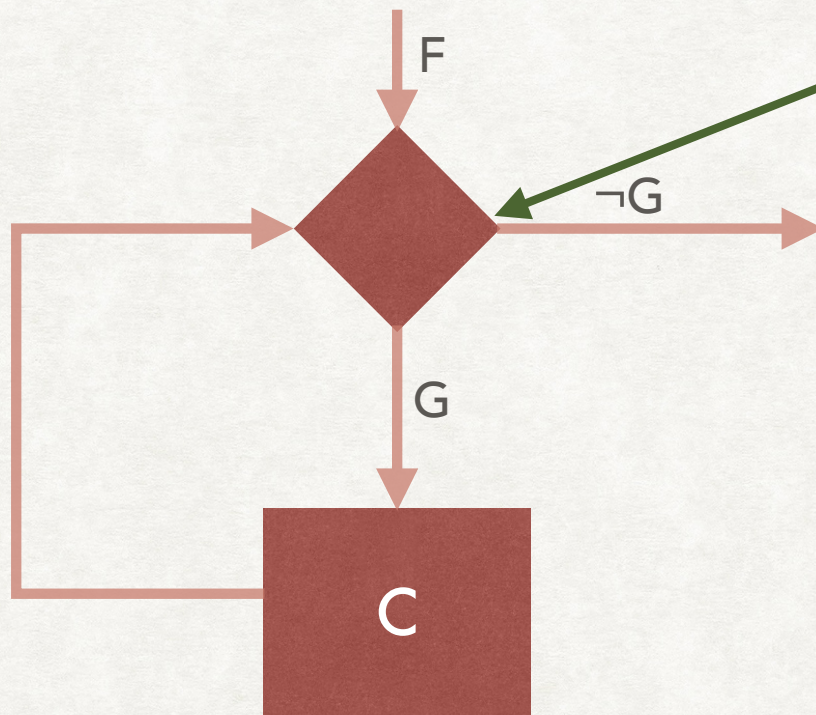


# STRONGEST POST-CONDITION

## WHILE LOOPS

- How to find  $sp(F, \text{while}(G) \text{ do } c)$ ?

Let us collect all states possible at the end of any iteration



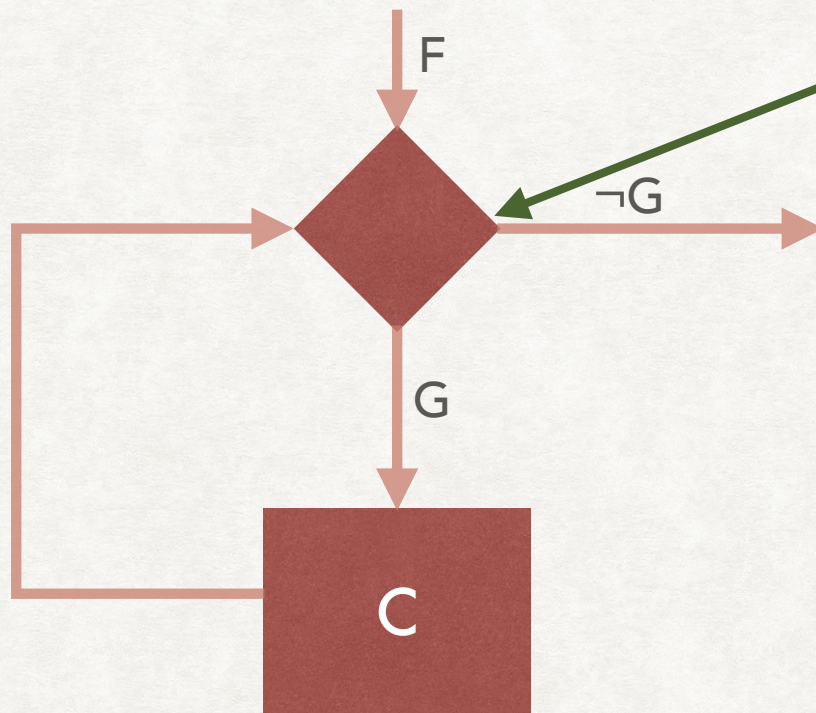
Iteration $i$	States possible at iteration $i$
0	
1	
2	
...	...

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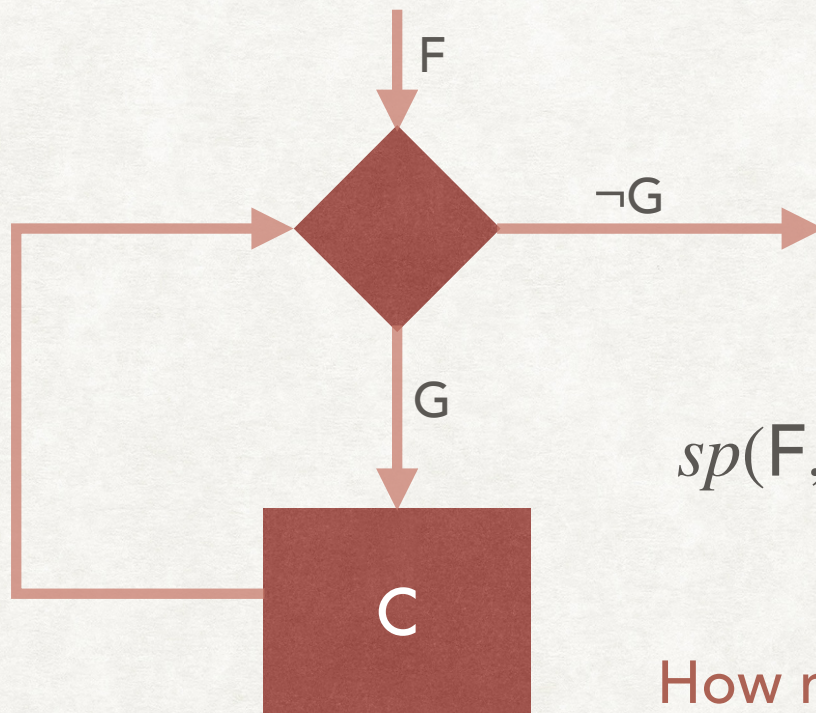


Iteration $i$	States possible at iteration $i$
0	$F$
1	$sp(F \wedge G, c)$
2	$sp(sp(F \wedge G, c) \wedge G, c)$
...	...

# STRONGEST POST-CONDITION

## WHILE LOOPS

- How to find  $sp(F, \text{while}(G) \text{ do } c)$ ?



$$F^0 = F$$

$$F^k = sp(F^{k-1} \wedge G, c)$$

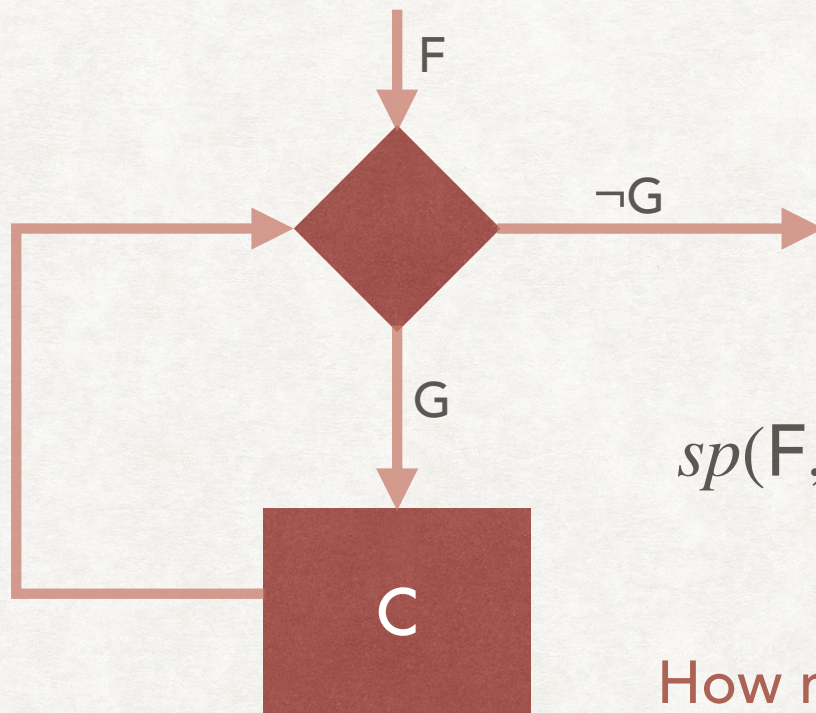
$$sp(F, \text{while}(G) \text{ do } c) \triangleq \bigvee_{k=0}^{\infty} F^k \wedge \neg G$$

How many  $F^k$  should be calculated?

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## WHILE LOOPS

- How to find  $sp(F, \text{while}(G) \text{ do } c)$ ?



$$F^0 = F$$

$$F^k = sp(F^{k-1} \wedge G, c)$$

$$sp(F, \text{while}(G) \text{ do } c) \triangleq \bigvee_{k=0}^{\infty} F^k \wedge \neg G$$

How many  $F^k$  should be calculated?

$$\text{Until } F^k \rightarrow \bigvee_{i=0}^{k-1} F^i$$



# WHILE LOOPS

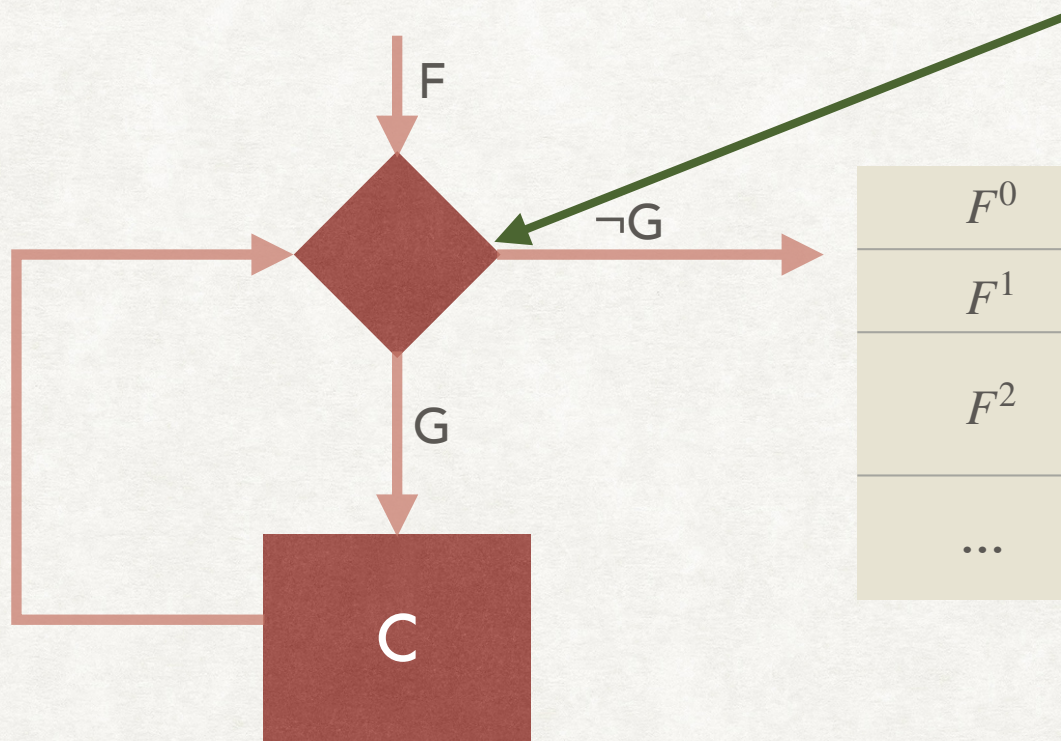
## EXAMPLES

- $sp(x < 0, \text{while}(x < 10) \text{ do } x := x + 1; ) \triangleq ???$

# WHILE LOOPS

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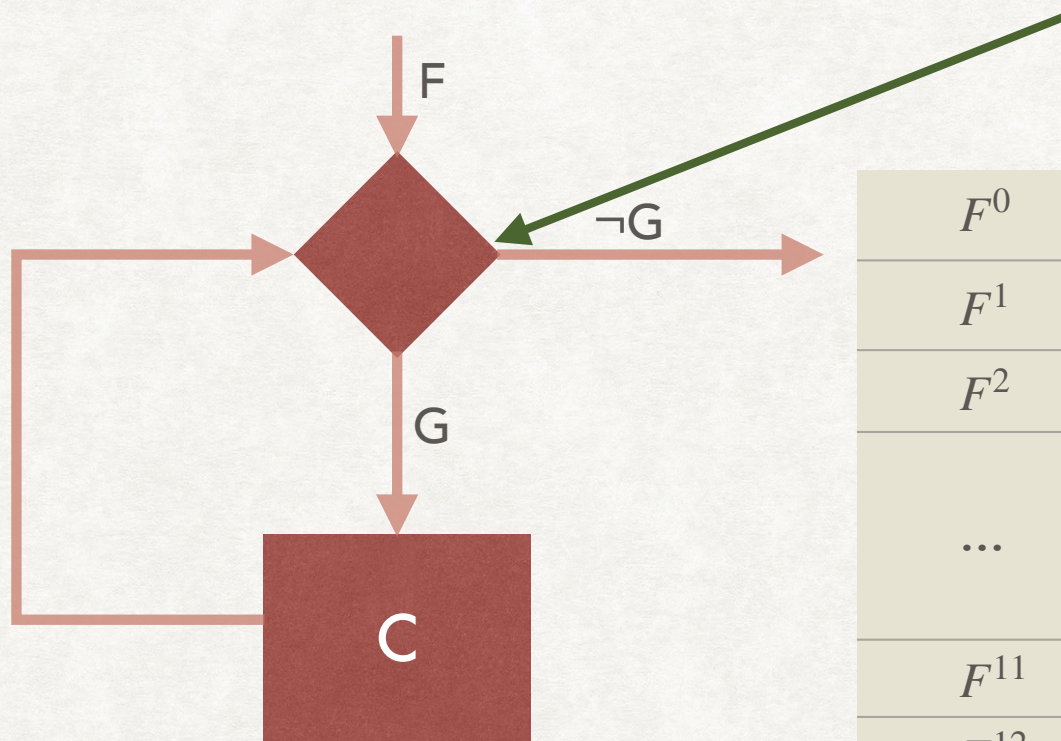


$F^0$	$F$
$F^1$	$sp(F \wedge G, c)$
$F^2$	$sp(sp(F \wedge G, c) \wedge G, c)$
...	...

# WHILE LOOPS

## EXAMPLES

- $sp(x < 0, \text{while}(x < 10) \text{ do } x := x + 1; ) \triangleq ???$

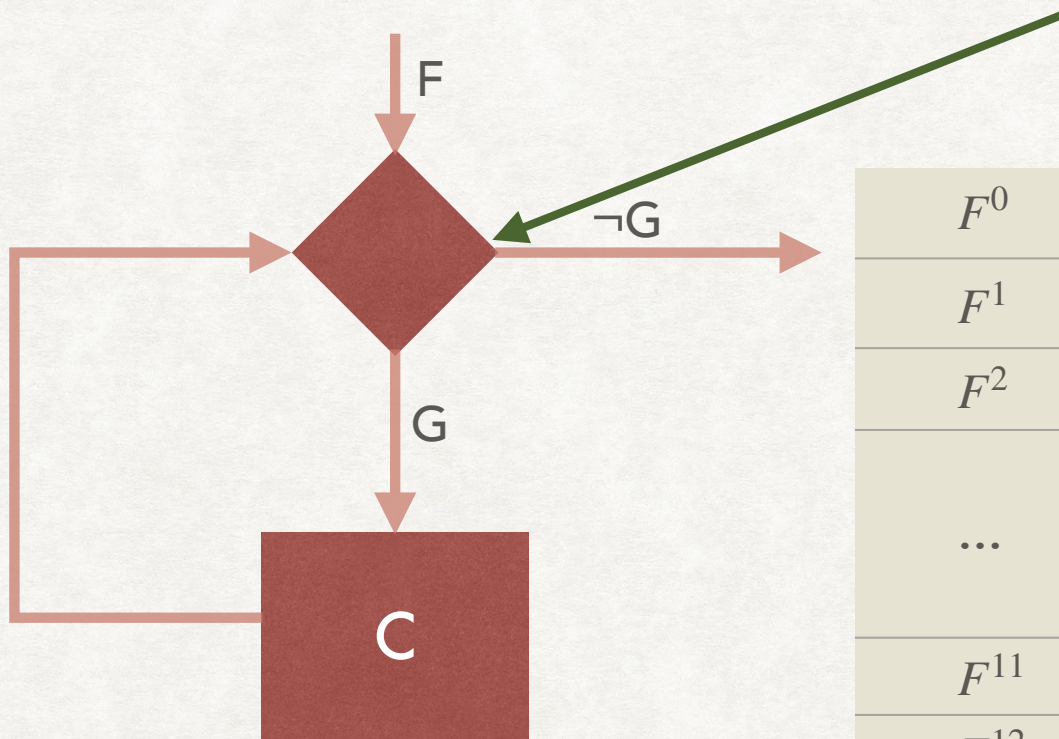


$F^0$	
$F^1$	
$F^2$	
...	...
$F^{11}$	
$F^{12}$	

# WHILE LOOPS

## EXAMPLES

- $sp(x < 0, \text{while}(x < 10) \text{ do } x := x + 1; ) \triangleq ???$



$F^0$	$x < 0$
$F^1$	$x < 1$
$F^2$	$x < 2$
...	...
$F^{11}$	$x < 11$
$F^{12}$	$x < 11$

# WHILE LOOPS

## EXAMPLES

- $sp(x < 0, \text{while}(x < 10) \text{ do } x := x + 1; ) \triangleq x \geq 10 \wedge x < 11$

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- $sp(x < 0, \text{while}(x < 10) \text{ do } x := x + 1; ) \triangleq x \geq 10 \wedge x < 11$
- $sp(x > 0, \text{while}(x < 10) \text{ do } x := x + 1; ) \triangleq ???$

# WHILE LOOPS

## EXAMPLES

- $sp(x < 0, \text{while}(x < 10) \text{ do } x := x + 1; ) \triangleq x \geq 10 \wedge x < 11$
- $sp(x > 0, \text{while}(x < 10) \text{ do } x := x + 1; ) \triangleq x \geq 10$

# WHILE LOOPS

## EXAMPLES

- $sp(x < 0, \text{while}(x < 10) \text{ do } x := x + 1; ) \triangleq x \geq 10 \wedge x < 11$
- $sp(x > 0, \text{while}(x < 10) \text{ do } x := x + 1; ) \triangleq x \geq 10$
- $sp(x > 0, \text{while}(x > 0) \text{ do } x := x + 1; ) \triangleq ???$



# WHILE LOOPS

## EXAMPLES

- $sp(x < 0, \text{while}(x < 10) \text{ do } x := x + 1; ) \triangleq x \geq 10 \wedge x < 11$
- $sp(x > 0, \text{while}(x < 10) \text{ do } x := x + 1; ) \triangleq x \geq 10$
- $sp(x > 0, \text{while}(x > 0) \text{ do } x := x + 1; ) \triangleq \perp$

# STRONGEST POST-CONDITION AND VERIFICATION

- Given a program  $c$  with assertions, then  $c$  is safe if  $sp(\text{error} = 0, c) \rightarrow \text{error} = 0$  is valid.
- The Hoare Triple  $\{P\}c\{Q\}$  is valid if  $sp(P, c) \rightarrow Q$  is valid.
- Do we have a decidable procedure for  $sp$ ?
  - No, due to the (potentially) unbounded computation required for while loops.
  - What is an example of  $F$  and  $c$  such that  $sp(F, c)$  requires unbounded computation?

# COMPLETE EXAMPLE

1: `assume(i = 0 ∧ n ≥ 0);`

2: `while(i < n) do`

3: `i := i + 1;`

4: `assert(i = n);`

$sp(error = 0, 1 :) \equiv error = 0 \wedge i = 0 \wedge n \geq 0$

$sp(error = 0 \wedge i = 0 \wedge n \geq 0 \wedge i < n, 3 :) \equiv \exists i, n, error. error = 0 \wedge i = 0 \wedge$   
 $n \geq 0 \wedge i < n \wedge i' = i + 1 \wedge frame(i)$

# COMPLETE EXAMPLE

1: `assume(i = 0 ∧ n ≥ 0);`

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$sp(error = 0 \wedge i = 0 \wedge n \geq 0 \wedge i < n, 3 :) \equiv \exists i, n, error. error = 0 \wedge i = 0 \wedge$   
 $n \geq 0 \wedge i < n \wedge i' = i + 1 \wedge frame(i)$   
 $\equiv error = 0 \wedge i = 1 \wedge n > 0$

$sp(error = 0 \wedge i = 1 \wedge n \geq 1 \wedge i < n, 3 :) \equiv error = 0 \wedge i = 2 \wedge n \geq 2$



# COMPLETE EXAMPLE

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2: `while(i < n) do`

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$sp(error = 0, 1 :) \equiv error = 0 \wedge i = 0 \wedge n \geq 0$

$sp(error = 0 \wedge i = 0 \wedge n \geq 0, 2 :) \equiv error = 0 \wedge ((i = 0 \wedge n \geq 0) \vee (i = 1 \wedge n \geq 1) \vee (i = 2 \wedge n \geq 2) \vee \dots) \wedge i \geq n$

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$sp(error = 0, 1 :) \equiv error = 0 \wedge i = 0 \wedge n \geq 0$

$sp(error = 0 \wedge i = 0 \wedge n \geq 0, 2 :) \equiv error = 0 \wedge ((i = 0 \wedge n \geq 0) \vee (i = 1 \wedge n \geq 1) \vee (i = 2 \wedge n \geq 2) \vee \dots) \wedge i \geq n$   
 $\equiv error = 0 \wedge i = n$

$sp(error = 0 \wedge i = n, 4 :) \equiv error = 0 \wedge i = n \vee \exists V. error = 0 \wedge i = n \wedge i \neq n$   
 $\equiv error = 0 \wedge i = n$   
 $\Rightarrow error = 0$

# SOUNDNESS AND COMPLETENESS

- Verification Problem: For a given program  $c$ , is  $(Error, c')$  reachable for some  $c'$ ?
  - A program  $c$  is safe if  $(Error, c')$  is not reachable for any  $c'$ .
  - Denoted as  $\vdash c$  is safe.
- Strongest Post-condition based Verification Procedure
  - $c$  is safe if  $sp(error = 0, c) \Rightarrow error = 0$ .
  - Denoted as  $\models c$  is safe.
- Soundness: Does  $\models c$  is safe  $\Rightarrow \vdash c$  is safe?
- Completeness: Does  $\vdash c$  is safe  $\Rightarrow \models c$  is safe?

# SOUNDNESS AND COMPLETENESS

## STRONGEST POST-CONDITION

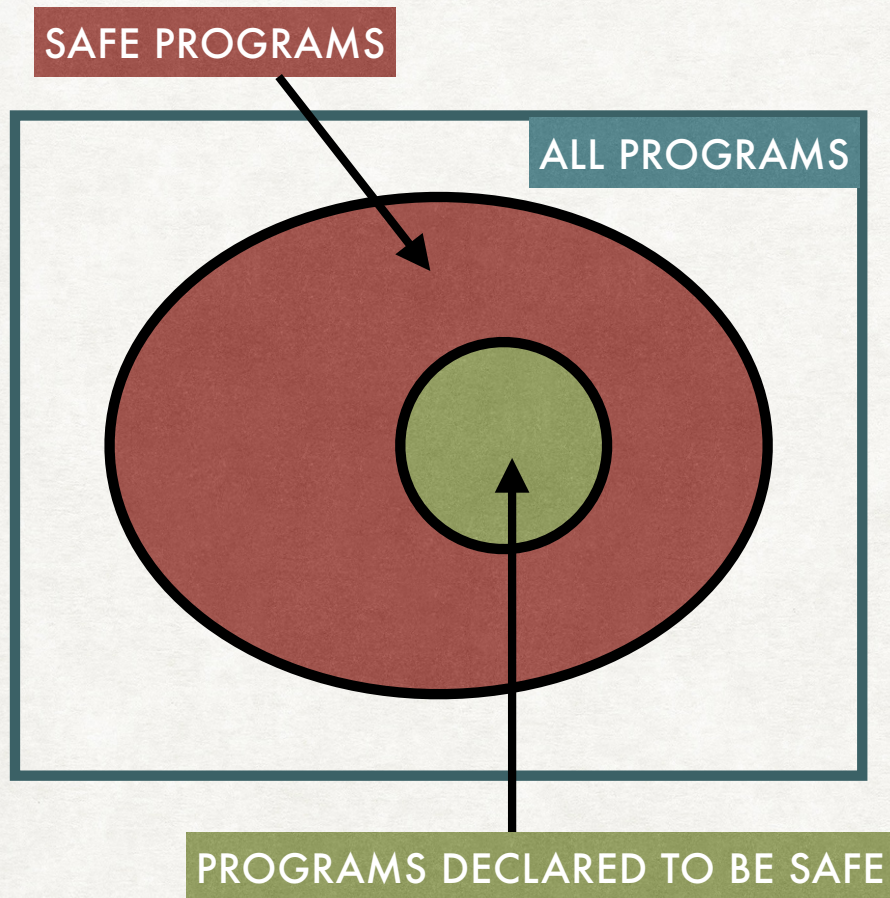
- Is the Strongest Post-condition based verification procedure sound?
  - Yes, by construction.
- Is the Strongest Post-condition based verification procedure complete?
  - Relatively Complete.
  - Relies on validity (modulo theory) of an FOL formula.
  - If the FOL theory is incomplete, then there could be valid formulae in the theory, which are not valid modulo the axioms. E.g. Peano Arithmetic.
  - However, the Theory of Reals (and Theory of Integers) are both complete, and hence the procedure is also complete.



# SOUNDNESS AND COMPLETENESS

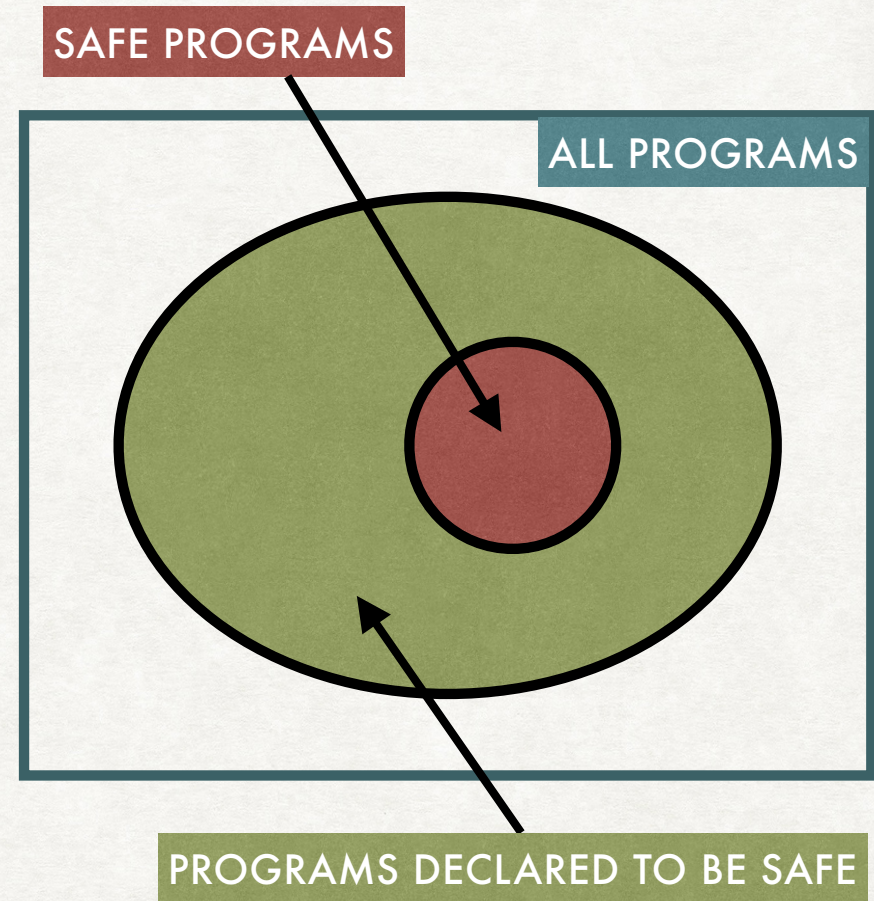
- What is an example of a trivial sound but in-complete verification procedure?
  - Simply say all programs are un-safe
- What is an example of a trivial un-sound but complete verification procedure?
  - Simply say all programs are safe

# SOUNDNESS AND COMPLETENESS



Sound, In-complete  
Verification Procedure

Examples : Strongest Post-  
condition, Weakest Pre-condition,  
Hoare Logic...



Un-sound, complete  
Verification Procedure

Examples : Testing, Fuzzing,  
Bounded Model-Checking...

# ISSUES WITH SP

- Loops can lead to unbounded computation.
- *sp* computation for assignments is tedious and inefficient.  
Requires quantifier elimination after every assignment statement.
- Ideally, we want a syntactic, linear-time method for computing the FOL formula.