

ANNOUNCEMENT

- Course Project
 - Start working on the project proposal (Due Date: Oct 14).
 - Explore sub-areas, Case studies, Study advanced verification tools,...
 - We will have one-on-one meetings this Friday to discuss plans.
 - Poll to pick a slot will be out by end of the day.

COMPACTNESS OF FOL

- An infinite set of FOL formulae is simultaneously satisfiable if and only if every finite subset is satisfiable.
- Due to compactness, many interesting properties cannot be expressed in First-order Logic.
- In particular, **transitive closure** cannot be expressed in FOL
 - Has major implications on using FOL for program verification!

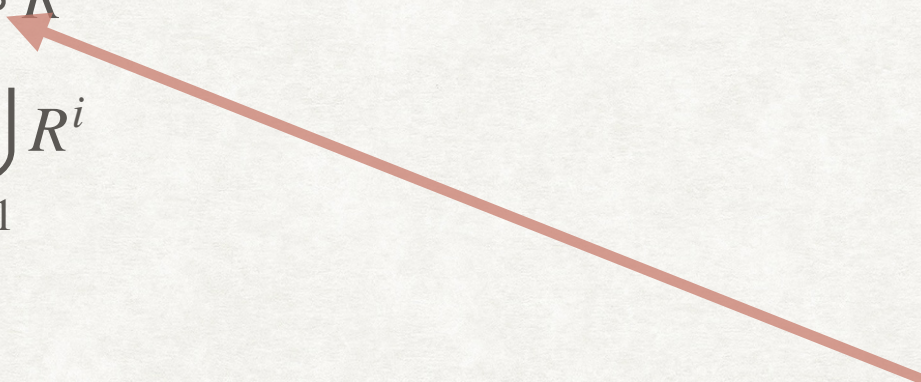
TRANSITIVE CLOSURE

- Given a binary relation R , its transitive closure R^* is defined as follows:

- $R^1 = R$

- $R^k = R \circ R^{k-1}$

- $R^* = \bigcup_{i \geq 1} R^i$


$$P \circ Q = \{(x, z) \mid (x, y) \in P \wedge (y, z) \in Q\}$$

TRANSITIVE CLOSURE IN FOL

- Let a binary predicate r represent the relation R , and let binary predicate T represent R^* .
- $F = \forall x, z. T(x, z) \leftrightarrow (r(x, z) \vee (\exists y. r(x, y) \wedge T(y, z)))$
 - Does this formula not represent transitive closure?!
 - Seems to directly encode $R^k = R \circ R^{k-1}$?

TRANSITIVE CLOSURE IN FOL

$$F = \forall x, z. T(x, z) \leftrightarrow (r(x, z) \vee (\exists y. r(x, y) \wedge T(y, z)))$$

- Consider following interpretation I :
 - $D_I = \{A, B\}$
 - $\alpha_I[r] = \{(A, A) \mapsto \text{true}, (B, B) \mapsto \text{true}, (A, B) \mapsto \text{false}, (B, A) \mapsto \text{false}\}$
 - $\alpha_I[T] = \{(A, A) \mapsto \text{true}, (B, B) \mapsto \text{true}, (A, B) \mapsto \text{true}, (B, A) \mapsto \text{true}\}$
- Transitive closure of r is r itself, but $I \models F$!
- $F : \forall x. \forall z. (T(x, z) \leftrightarrow (r(x, z) \vee \exists y. r(x, y) \wedge T(y, z)))$ does not represent transitive closure

COMPACTNESS OF FOL AND TRANSITIVE CLOSURE - I

- Compactness: An infinite set of FOL formulae is simultaneously satisfiable if and only if every finite subset is satisfiable.
- Assume that Γ is a FOL formula which encodes the transitive closure T of relation r .
- Let $\Psi_n(x, y)$ encode that there is no path of length n in the relation r between x and y .
 - $\Psi_1(x, y) = \neg r(x, y)$
 - $\Psi_n(x, y) = \neg \exists x_1, \dots, x_{n-1} . r(x, x_1) \wedge \dots \wedge r(x_{n-1}, y)$

COMPACTNESS OF FOL AND TRANSITIVE CLOSURE - II

- Consider the following infinite set of FOL formulae:
 $\Gamma' = \{\Gamma, T(a, b), \Psi_1(a, b), \Psi_2(a, b), \dots\}$
- Note that Γ' is unsatisfiable. Why?
 - Since Γ is a correct encoding of Transitive Closure, $T(a, b)$ asserts that there is some path.
 - But all the Ψ s assert that there is no path of any length.

COMPACTNESS OF FOL AND TRANSITIVE CLOSURE - III

- However, consider any finite subset of $\Gamma' = \{\Gamma, T(a, b), \Psi_1(a, b), \Psi_2(a, b), \dots\}$.
- If it does not contain Γ or $T(a, b)$, then it is clearly satisfiable. (Why?)
- If it contains both Γ and $T(a, b)$, it will not contain $\Psi_i(a, b)$ for some i . Hence, it is again satisfiable.
- Thus, every finite subset of Γ' is satisfiable, and hence by the compactness of FOL, Γ' should also be satisfiable.
 - This leads to contradiction, thus showing that there cannot exist Γ which can encode transitive closure.