ANNOUNCEMENT

- Course Project
  - Start working on the project proposal (Due Date: Oct 14).
  - Explore sub-areas, Case studies, Study advanced verification tools,…
  - We will have one-on-one meetings this Friday to discuss plans.
  - Poll to pick a slot will be out by end of the day.

COMPACTNESS OF FOL

- An infinite set of FOL formulae is simultaneously satisfiable if and only if every finite subset is satisfiable.
- Due to compactness, many interesting properties cannot be expressed in First-order Logic.
- In particular, transitive closure cannot be expressed in FOL
  - Has major implications on using FOL for program verification!

TRANSITIVE CLOSURE

- Given a binary relation R, its transitive closure R^* is defined as follows:
  - $R^1 = R$
  - $R^k = R \circ R^{k-1}$
  - $R^* = \bigcup_{i \geq 1} R^i$

$P \circ Q = \{(x, z) \mid (x, y) \in P \land (y, z) \in Q\}$

TRANSITIVE CLOSURE IN FOL

- Let a binary predicate r represent the relation R, and let binary predicate T represent R^*.
- $F = \forall x, z \,.\, T(x,z) \leftrightarrow (r(x,z) \vee (\exists y \,.\, r(x,y) \wedge T(y,z)))$
  - Does this formula not represent transitive closure?!
  - Seems to directly encode $R^k = R \circ R^{k-1}$?

TRANSITIVE CLOSURE IN FOL

$$F = \forall x, z\,.\,T(x,z) \leftrightarrow (r(x,z) \lor (\exists y\,.\,r(x,y) \land T(y,z)))$$

- Consider following interpretation I:
  - $D_I = \{A, B\}$
  - $\alpha_I[r] = \{(A,A) \mapsto true, (B,B) \mapsto true, (A,B) \mapsto false, (B,A) \mapsto false\}$
  - $\alpha_I[T] = \{(A,A) \mapsto true, (B,B) \mapsto true, (A,B) \mapsto true, (B,A) \mapsto true\}$
- Transitive closure of r is r itself, but $I \vDash F$!
- $F : \forall x\,.\,\forall z\,.\,(T(x,z) \leftrightarrow (r(x,z) \lor \exists y\,.\,r(x,y) \land T(y,z)))$ does not represent transitive closure

COMPACTNESS OF FOL AND TRANSITIVE CLOSURE - I

- Compactness: An infinite set of FOL formulae is simultaneously satisfiable if and only if every finite subset is satisfiable.
- Assume that Γ is a FOL formula which encodes the transitive closure T of relation r.
- Let $\Psi_n(x, y)$ encode that there is no path of length n in the relation r between x and y.
  - $\Psi_1(x, y) = \neg r(x, y)$
  - $\Psi_n(x, y) = \neg \exists x_1, \ldots, x_{n-1} . r(x, x_1) \wedge \ldots r(x_{n-1}, y)$

COMPACTNESS OF FOL AND TRANSITIVE CLOSURE - II

- Consider the following infinite set of FOL formulae:
  $\Gamma' = \{\Gamma, T(a,b), \Psi_1(a,b), \Psi_2(a,b), \ldots\}$
- Note that Γ' is unsatisfiable. Why?
  - Since Γ is a correct encoding of Transitive Closure, $T(a,b)$ asserts that there is some path.
  - But all the Ψs assert that there is no path of any length.

COMPACTNESS OF FOL AND TRANSITIVE CLOSURE - III

- However, consider any finite subset of $\Gamma' = \{\Gamma, T(a,b), \Psi_1(a,b), \Psi_2(a,b), \ldots\}$.
- If it does not contain Γ or $T(a,b)$, then it is clearly satisfiable. (Why?)
- If it contains both Γ and $T(a,b)$, it will not contain $\Psi_i(a,b)$ for some i. Hence, it is again satisfiable.
- Thus, every finite subset of Γ' is satisfiable, and hence by the compactness of FOL, Γ' should also be satisfiable.
  - This leads to contradiction, thus showing that there cannot exists Γ which can encode transitive closure.