SATISFIABILITY AND VALIDITY

- A FOL formula F is satisfiable if there exists an interpretation I such that $I \vDash F$.
    - If no such interpretation exists, then it is unsatisfiable
- A FOL formula F is valid if for all interpretations I, $I \vDash F$
- F is valid iff $\neg F$ is unsatisfiable.

SATISFIABILITY AND VALIDITY

EXAMPLES

- Is the formula $\forall x . \exists y . p(x, y)$ satisfiable?
  - Yes. A satisfying interpretation:
    $I = (\{A\}, < p \mapsto \{(A, A) \mapsto true\} >)$
- Is the formula $\forall x . \exists y . p(x, y)$ valid?
  - No. A falsifying interpretation:
    $I = (\{A\}, < p \mapsto \{(A, A) \mapsto false\} >)$
- Is the formula $(\forall x . p(x)) \to (\exists y . p(y))$ valid?
- Is the formula $\forall x . (p(x) \to (\exists y . p(y)))$ valid?
  - What about $\forall x . (p(x) \to (\forall y . p(y)))$?

DECISION PROCEDURE FOR VALIDITY

- Semantic Argument Method
  - Deductive Approach
  - Proof by Contradiction
  - Assume that a falsifying interpretation exists.
  - Use proof rules to deduce more facts.
  - Find contradictory facts in each branch (also called closing the branch).
- Proof rules for negation, conjunction, disjunction, implication, iff carry over from Propositional logic

PROOF RULES

UNIVERSAL QUANTIFICATION

$$\frac{I \vDash \forall x \,.\, F}{I[x \mapsto v] \vDash F} \quad \text{For any } v \in D_I$$

PROOF RULES

UNIVERSAL QUANTIFICATION

$$\frac{I \models \forall x \,.\, F}{I[x \mapsto v] \models F}$$ For any $v \in D_I$

$$\frac{I \not\models \forall x \,.\, F}{I[x \mapsto v] \not\models F}$$ For a fresh $v \in D_I$

PROOF RULES

EXISTENTIAL QUANTIFICATION

$$\frac{I \models \exists x \,.\, F}{I[x \mapsto v] \models F}$$

For a fresh $v \in D_I$

PROOF RULES

EXISTENTIAL QUANTIFICATION

$$\frac{I \models \exists x \,.\, F}{I[x \mapsto v] \models F}$$ For a fresh $v \in D_I$

$$\frac{I \not\models \exists x \,.\, F}{I[x \mapsto v] \not\models F}$$ For any $v \in D_I$

PROOF RULES

CONTRADICTION

$$\frac{J \vDash p(s_1, \ldots, s_n) \quad K \nvDash p(t_1, \ldots, t_n) \quad J = I[\ldots] \quad K = I[\ldots] \quad \alpha_J[s_i] = \alpha_K[t_i] \text{ for all } i = 1, \ldots, n}{I \vDash \bot}$$

EXAMPLE

Prove that $(\forall x \,.\, p(x)) \rightarrow (\forall y \,.\, p(y))$ is valid

$$\frac{I \not\models (\forall x \,.\, p(x)) \rightarrow (\forall y \,.\, p(y))}{I \models (\forall x \,.\, p(x)) \qquad I \not\models (\forall y \,.\, p(y))}$$

EXAMPLE

Prove that $(\forall x \,.\, p(x)) \rightarrow (\forall y \,.\, p(y))$ is valid

$$\dfrac{I \not\models (\forall x \,.\, p(x)) \rightarrow (\forall y \,.\, p(y))}{I \models (\forall x \,.\, p(x)) \qquad \dfrac{I \not\models (\forall y \,.\, p(y))}{I[y \mapsto v] \not\models p(y)}\ \text{[for a fresh } v\text{]}}$$

EXAMPLE

Prove that $(\forall x \,.\, p(x)) \to (\forall y \,.\, p(y))$ is valid

$$\dfrac{I \not\models (\forall x \,.\, p(x)) \to (\forall y \,.\, p(y))}{I \models (\forall x \,.\, p(x)) \qquad I \not\models (\forall y \,.\, p(y))}$$

$$\dfrac{I \models (\forall x \,.\, p(x))}{I[x \mapsto v] \models p(x)} \qquad \dfrac{I \not\models (\forall y \,.\, p(y))}{I[y \mapsto v] \not\models p(y)} \quad \text{[for a fresh } v\text{]}$$

EXAMPLE

Prove that $(\forall x \,.\, p(x)) \rightarrow (\forall y \,.\, p(y))$ is valid

$$\dfrac{I \not\models (\forall x \,.\, p(x)) \rightarrow (\forall y \,.\, p(y))}{\dfrac{I \models (\forall x \,.\, p(x))}{I[x \mapsto v] \models p(x)} \qquad \dfrac{I \not\models (\forall y \,.\, p(y))}{I[y \mapsto v] \not\models p(y)}}$$

[for a fresh v]

CONTRADICTION

EXAMPLE - 2

Prove that $\exists x \,.\, F \rightarrow G \leftrightarrow (\forall x \,.\, F) \rightarrow (\exists x \,.\, G)$ is valid

EXAMPLE - 2

Prove that $\exists x\,.\,F \rightarrow G \leftrightarrow (\forall x\,.\,F) \rightarrow (\exists x\,.\,G)$ is valid

$$I \not\models \exists x\,.\,F \rightarrow G \leftrightarrow (\forall x\,.\,F) \rightarrow (\exists x\,.\,G)$$

EXAMPLE - 2

Prove that $\exists x . F \rightarrow G \leftrightarrow (\forall x . F) \rightarrow (\exists x . G)$ is valid

$$I \not\models \exists x . F \rightarrow G \leftrightarrow (\forall x . F) \rightarrow (\exists x . G)$$

$I \models \exists x . F \rightarrow G \quad I \not\models (\forall x . F) \rightarrow (\exists x . G)$ | $I \not\models \exists x . F \rightarrow G \quad I \models (\forall x . F) \rightarrow (\exists x . G)$

EXAMPLE - 2

Prove that $\exists x . F \rightarrow G \leftrightarrow (\forall x . F) \rightarrow (\exists x . G)$ is valid

$$I \not\models \exists x . F \rightarrow G \leftrightarrow (\forall x . F) \rightarrow (\exists x . G)$$

$I \models \exists x . F \rightarrow G$ $\qquad$ $I \not\models (\forall x . F) \rightarrow (\exists x . G)$

EXAMPLE - 2

Prove that $\exists x\,.\,F \rightarrow G \leftrightarrow (\forall x\,.\,F) \rightarrow (\exists x\,.\,G)$ is valid

$$I \not\models \exists x\,.\,F \rightarrow G \leftrightarrow (\forall x\,.\,F) \rightarrow (\exists x\,.\,G)$$

$$I \models \exists x\,.\,F \rightarrow G \qquad\qquad I \not\models (\forall x\,.\,F) \rightarrow (\exists x\,.\,G)$$

$$I[x \mapsto v] \models F \rightarrow G$$

EXAMPLE - 2

Prove that $\exists x . F \rightarrow G \leftrightarrow (\forall x . F) \rightarrow (\exists x . G)$ is valid

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$$I \models \exists x . F \rightarrow G \qquad I \not\models (\forall x . F) \rightarrow (\exists x . G)$$

$$I[x \mapsto v] \models F \rightarrow G$$

$$I[x \mapsto v] \not\models F \quad | \quad I[x \mapsto v] \models G$$

EXAMPLE - 2

Prove that $\exists x \,.\, F \rightarrow G \leftrightarrow (\forall x \,.\, F) \rightarrow (\exists x \,.\, G)$ is valid

$$I \not\models \exists x \,.\, F \rightarrow G \leftrightarrow (\forall x \,.\, F) \rightarrow (\exists x \,.\, G)$$

$$I \models \exists x \,.\, F \rightarrow G \qquad I \not\models (\forall x \,.\, F) \rightarrow (\exists x \,.\, G)$$

$$I[x \mapsto v] \models F \rightarrow G \qquad I \models (\forall x \,.\, F) \quad I \not\models (\exists x \,.\, G)$$

$$I[x \mapsto v] \not\models F \quad | \quad I[x \mapsto v] \models G$$

EXAMPLE - 2

Prove that $\exists x . F \rightarrow G \leftrightarrow (\forall x . F) \rightarrow (\exists x . G)$ is valid

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$$I[x \mapsto v] \models F \rightarrow G \qquad I \models (\forall x . F) \quad I \not\models (\exists x . G)$$

$$I[x \mapsto v] \not\models F \quad | \quad I[x \mapsto v] \models G \quad I[x \mapsto v] \models F$$

EXAMPLE - 2

Prove that $\exists x . F \rightarrow G \leftrightarrow (\forall x . F) \rightarrow (\exists x . G)$ is valid

$$I \not\models \exists x . F \rightarrow G \leftrightarrow (\forall x . F) \rightarrow (\exists x . G)$$

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$$I[x \mapsto v] \models F \rightarrow G \qquad I \models (\forall x . F) \quad I \not\models (\exists x . G)$$

$$I[x \mapsto v] \not\models F \quad | \quad I[x \mapsto v] \models G \qquad I[x \mapsto v] \models F$$

CONTRADICTION

EXAMPLE - 2

Prove that $\exists x . F \to G \leftrightarrow (\forall x . F) \to (\exists x . G)$ is valid

$$I \not\models \exists x . F \to G \leftrightarrow (\forall x . F) \to (\exists x . G)$$

$$I \models \exists x . F \to G \qquad I \not\models (\forall x . F) \to (\exists x . G)$$

$$I[x \mapsto v] \models F \to G \qquad I \models (\forall x . F) \quad I \not\models (\exists x . G)$$

$$I[x \mapsto v] \not\models F \quad | \quad I[x \mapsto v] \models G \qquad I[x \mapsto v] \models F \quad I[x \mapsto v] \not\models G$$

EXAMPLE - 2

Prove that $\exists x . F \rightarrow G \leftrightarrow (\forall x . F) \rightarrow (\exists x . G)$ is valid

$$I \not\models \exists x . F \rightarrow G \leftrightarrow (\forall x . F) \rightarrow (\exists x . G)$$

$$I \models \exists x . F \rightarrow G \qquad\qquad I \not\models (\forall x . F) \rightarrow (\exists x . G)$$

$$I[x \mapsto v] \models F \rightarrow G \qquad\qquad I \models (\forall x . F) \quad I \not\models (\exists x . G)$$

$$I[x \mapsto v] \not\models F \;\mid\; I[x \mapsto v] \models G \qquad I[x \mapsto v] \models F \quad I[x \mapsto v] \not\models G$$

CONTRADICTION

MORE EXAMPLES

- Prove or disprove validity of following FOL formulae
  - $\forall x \,.\, F \rightarrow G \leftrightarrow (\exists x \,.\, F) \rightarrow (\forall x \,.\, G)$
  - $(\forall x \,.\, p(x)) \leftrightarrow \neg(\exists x \,.\, \neg p(x))$
  - $(\exists x \,.\, p(x)) \rightarrow (\forall y \,.\, p(y))$
  - $\exists x \,.\, (p(x) \rightarrow \forall y \,.\, p(y))$

DECIDABILITY OF VALIDITY OF FOL

- Church and Turing showed that it is undecidable to find whether a first-order formula is valid or not.
- But we have just seen the Semantic Argument-based decision procedure!
  - How to instantiate domain values in Proof rules for quantifiers?
  - What order should proof rules be applied in?
- The semantic argument-based method can be augmented to make the validity of FOL problem semi-decidable.
  - If the input formula is valid, then the method will halt and answer positive.
  - If the input formula is not valid, then the method may never halt.
  - More details in the BM book [Chapter 2, Section 2.7].

NORMAL FORMS OF FOL

- Negation Normal Form (NNF)
  - Should use only $\neg$, $\wedge$, $\vee$ as the logical connectives, and $\neg$ should only be applied to literals
  - $\neg(\forall x . F) \Leftrightarrow \exists x . \neg F$ and $\neg(\exists x . F) \Leftrightarrow \forall x . \neg F$

PRENEX NORMAL FORM

- A formula is in Prenex Normal Form (PNF) if all of its quantifiers appear at the beginning of the formula:
  - $Q_1 x_1 . \ldots Q_n x_n . F[x_1, \ldots, x_n]$, where F is quantifier-free and may have $x_1, \ldots, x_n$ as free variables.
- How to convert an arbitrary formula F to PNF?

1. First, convert F to NNF (call it F_1).
2. If two quantified variables in F_1 have the same name, then rename them to fresh variables (obtaining the formula F_2).
3. Remove all quantifiers in F_2 to obtain F_3.
4. Add all the removed quantifiers at the beginning of F_3, ensuring that if Q_j was in the scope of Q_i in F_2, then Q_i occurs before Q_j

PRENEX NORMAL FORM

EXAMPLE

$$F:\ \forall x.\ \neg(\exists y.\ p(x,y)\ \wedge\ p(x,z))\ \vee\ \exists y.\ p(x,y)$$

STEP-1

$$F_1:\ \forall x.\ (\forall y.\ \neg p(x,y)\ \vee\ \neg p(x,z))\ \vee\ \exists y.\ p(x,y)$$

STEP-2

$$F_2:\ \forall x.\ (\forall y.\ \neg p(x,y)\ \vee\ \neg p(x,z))\ \vee\ \exists w.\ p(x,w)$$

STEP-3

$$F_3:\ \neg p(x,y)\ \vee\ \neg p(x,z)\ \vee\ p(x,w)$$

STEP-4

$$\forall x.\ \forall y.\ \exists w.\ \neg p(x,y)\ \vee\ \neg p(x,z)\ \vee\ p(x,w)$$